THEORY OF MAXWELLIAN CIRCUITS AND ITS APPLICATIONS TO MULTI WIRE SYSTEMS

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Abstract
A Maxwellian circuit is a direct circuit simulation of microwave components, such as small patches, gaps, bends, stubs etc, connected by a thin wire structure. A Maxwellian circuit contains resistances, inductances, capacitances, and voltage and current dependent sources, which can be solved for voltages and currents by the conventional circuit equations. The solutions of the Maxwellian circuit of a particular structure must be the same as those obtained by rigorous solutions of the Maxwell's equations of the original structure. For each point on a Maxwellian circuit there is a corresponding point in the original structure, and the voltage and current at that point are equal to those at the corresponding point on the original structure. This paper lays the foundation of Maxellian circuit by theorems of its existence and uniqueness, and gives the procedure of finding the circuit components of single and multi transmission line systems.

Keywords: Maxwellian circuits, equivalent circuits, circuit simulation, microwave structures, microstrip circuits.

Introduction
Circuit simulation of solutions to Maxwell's equations are important in VLSI, packaging and interconnects, and microwave/mm-wave circuit designs because there is a need for the combined use of solutions of Maxwell's equations and circuit analyses in those disciplines. Recently, it is shown that for each Maxwell's integral equation of the thin wire type, there exists an equivalent differential equation, the solutions of which are identical to those of the integral equation, if identical terminal boundary conditions are applied to the integral equation and the differential equation [1, 2]. When the differential equation is written in terms voltages and currents on the wire, we can get the equivalent circuit to the integral equation, whose solutions are identical to the integral equation or solution the Maxwell's equations. We name such equivalent circuit as a "Maxwellian circuit". In this paper, we shall also extend the concept of Maxwellian circuit to multi wire systems.

Basic Theorems
The theory of Maxwellian circuits is based on the following two theorems:

(I) Existence Theorem
For every integral equation of the thin wire type,

$$\int_{-L_a}^{L_b} G(\ell,\ell')\ \mathrm{I}(\ell')\ d\ell' = \mathrm{A}\sin k\ell + \mathrm{B}\cos k\ell - \frac{jV_0}{2Z_0}\sin k|\ell| \qquad (1)$$

where $G(l,l')$ is the kernel of the integral equation, l and l' are points along the wire, $I(l')$ is the current on the wire, Vo is the driving voltage located at a point $l = 0$, A and B are constants to be determined by the boundary condition of the current at both ends of the wire, there exists a differential equation of the form,

$$DI = \frac{d^2 I}{d\ell^2} + U(\ell)\frac{dI}{d\ell} + T(\ell)I = 0 \tag{2}$$

such that the solution of the differential equation is identical to the integral equation. The coefficients U and T are found by substituting two solutions of (1) into (2) and solve the resulting equations. The two solutions of the integral equation (1) may be obtained by applying two different boundary conditions at the terminals $l = -L_a$ and $l = L_b$.

(II) Uniqueness Theorem

Among the differential operators there is only one, which is independent of the boundary conditions at either ends of the wire structure.

Proofs of the Theorems

The proofs of the theorems are given in [1, 2].

The Circuit Representations

The second order differential equation (2) may be represented by two first order equations of voltages and currents,

$$\frac{dV}{d\ell} = -j\omega L(\ell)I + \alpha(\ell)V \tag{3a}$$

$$\frac{dI}{d\ell} = -j\omega C(\ell)V + \beta(\ell)I \tag{3b}$$

where the $L(l)$, $C(l)$, $\alpha(l)$ and $\beta(l)$ are circuit elements of a non-uniform transmission line. These equations are different from the conventional non-uniform transmission line equations by the addition of dependent source α and β. Without the dependent sources, the pair of first order differential equations is not the most general, so they may not be able to represent the second order differential equation of (2). Furthermore, the conventional transmission line equations are derived from Kirchhoff's laws [3, 4], which happen to be correct for uniform lines. To derive the non-uniform transmission line equations from Maxwell's equations, we need to start from the following equations,

$$\nabla V = -j\omega \mathrm{A} - \mathbf{E} \tag{4a}$$

$$\nabla \cdot \mathbf{A} = -j\omega\mu \in V \tag{4b}$$

where V is the scalar potential and **A** is the vector potential. If we use the familiar circuit approximation of **A** by,

$$\mathrm{A} \cdot \hat{\ell} = A_\ell = LI(\ell) \tag{5}$$

where $\hat{\ell}$ represents a unit vector along ℓ. Along the line, the equations in (4) become

$$\frac{dV}{d\ell} = -j\omega A_\ell \tag{6a}$$

$$\frac{d(L\mathrm{I})}{d\ell}=\frac{dL}{d\ell}\mathrm{I}+L\frac{d\mathrm{I}}{d\ell}=-j\omega\mu\in V\ or\ \frac{d\mathrm{I}}{d\ell}=\frac{-j\omega\mu\in}{L}V-\frac{dL}{Ld\ell}\mathrm{I} \quad (6b)$$

where the second equation is different from the conventional equation of,

$$\frac{d\mathrm{I}}{d\ell}=-j\omega C(\ell)V \quad (7)$$

which ignores the effect of changing line inductance. So, the text book equations for non-uniform transmission line is not correct [3, 4]. It is really quite astonishing that authoritative text books on electromagnetic theory would derive the transmission line equations from Kirchhoff's laws and would stick to it when the line became non-uniform. In order to derive the transmission line equations from Maxwell's equations, we should use the most general first order equations of (3). To find the parameters of the differential equations, we need two solutions of the current obtained from solutions of Maxwell's equations subject to different terminating conditions. For wire structures, the method of moments (MoM) is the most convenient for such purpose. Converting the first order equations to a second order differential equation is immediate and it should be identical to eq. (2) by the uniqueness theorem. The solutions of (3) should be identical to that of the integral equation, since we have obtained the parameters from the integral equation solutions. Solving the differential equation is just a reversing process of obtaining the solutions of the integral equations, from which the parameters are found. Therefore, the equivalent circuit of eq.(3) is Maxwellian because its solutions of currents and voltages are identical to those obtained by solving the Maxwell's equations of the original structure. Furthermore, the circuit topology is such that there is a one to one correspondence between a point of the equivalent circuit and that of the original structure.

The Missing Links

There is no doubt that by adding the dependent sources into the transmission line equations, we are able to find circuits, which reproduce the solutions of Maxwell's equations. So far, we have confined our reasoning to the mathematical interpretations of the dependent sources. Actually, dependent sources are familiar circuit components to circuit engineers in active circuits. The novelty here is that we are using them in transmission lines, which are thought to be passive. Is a transmission line really passive? It is so, if it were a closed system, such as a coaxial line. An element of an open line such as a microstrip may not be so easily classified. It is passive to the transmission system if it radiates and takes energy away from the system. It is active to the system if it receives energy from the surrounding. So, an open transmission line element could be either active or passive depending on the direction of the current flow, a role, which cannot be fulfilled by a pure passive or active element, but a dependent source fits in very naturally. Therefore, the dependent sources are the missing links between Kirchhoff's laws and Maxwell's equations, and they are part of the Maxwellian formulation. The traditional transmission line equations are based on Kirchhoff's laws, which are the conservation of current and the vanishing of the loop integral of the electric field. In Maxwell's equations, the integration of the electric field between two points is not independent of the path. It is the integration of the sum of the electric field and time derivative of the vector potential between two points that is independent of the path, i.e.,

$$\mathrm{V}_\ell=\int_0^\ell(-j\omega\mu\mathrm{A}\cdot\hat{\ell}-\mathbf{E}\cdot\hat{\ell})d\ell \quad (8)$$

to be independent of the path. And, that the summation of all the current going into a junction at a particular time should not always vanish, but the charge must be conserved at all time. The dependent sources in eq.(3) liberate them from Kirchhoff's laws. Our next task is to merge the equations with Maxwell's equations. To do that we determine the parameters L, C, α and β from the solutions of Maxwell's equations. That alone may not guarantee their consistency, however the uniqueness theorem does. Because when that set of parameters can produce all solutions of Maxwell's equation of the original system, regardless of how one loads it, it has to be consistent with the Maxwell's equations of the system. That is why we call the circuit "Maxwellian".

Equivalent Differential Equations for Multi-wires

The extension of the above theorems to multi wires is important and challenging, in that multi-wire configurations appear frequently in packaging, interconnect of IC's, and that to go from coupled integral equations to coupled differential equations is not a known art. Without the loss of generality, we shall limit the following derivations to two coupled straight wires, since the extension to N wire is immediate. The Hallen's type coupled integral equations for a two wire system is:

$$\boldsymbol{L}_{11}\mathrm{I}_1+\boldsymbol{L}_{12}\mathrm{I}_2=A_1\sin k\ell_1+B_1\cos k\ell_1-\frac{jV_0}{2Z_0}\sin k|\ell_1| \tag{9a}$$

$$\boldsymbol{L}_{21}\mathrm{I}_1+\boldsymbol{L}_{22}\mathrm{I}_2=A_2\sin k\ell_2+B_2\cos k\ell_2 \tag{9b}$$

assuming only one wire is being driven. The independent variables l_1 and l_2 represent points on wire 1 and 2 respectively, and operators $\boldsymbol{L}_{ij}$ represent the integral operator with field point on wire i and source point on wire j, for example,

$$\boldsymbol{L}_{ij}\mathrm{I}_j=\int_{-L_{ja}}^{L_{jb}}G(\ell_i/\ell_j')\,\hat{\ell}_i\cdot\hat{\ell}_j\,\mathrm{I}(\ell_j')d\ell_j' \tag{10}$$

We can eliminate I   to get the integral equation of I_1. The result is:

$$[\boldsymbol{L}_{11}-\boldsymbol{L}_{12}\boldsymbol{L}_{22}^{-1}\boldsymbol{L}_{21}]\mathrm{I}_1=A_1\sin k\ell_1+B_1\cos k\ell_1-\frac{jV_0}{2Z_0}\sin k|\ell_1|-A_2\boldsymbol{L}_{12}\boldsymbol{L}_{22}^{-1}\sin k\ell_2-B_2\boldsymbol{L}_{12}\boldsymbol{L}_{22}^{-1}\cos k\ell_2 \tag{11}$$

which can be abbreviated by,

$$\boldsymbol{L}_{01}\mathrm{I}_1=A_1\sin k\ell_1+B_1\cos k\ell_1-\frac{jV_0}{2Z_0}\sin k|\ell_1|-A_2f_2(\ell_1)-B_2g(\ell)\delta \tag{12}$$

where $\mathrm{f}_2(\ell_1)$ and $\mathrm{g}_2(\ell_1)$ are functions of the geometry of the system but not of the excitation or the boundary conditions on ℓ_1. Together with the constants A_2 and B_2, f_2 and g_2 represent the effects of the boundary conditions of wire 2 on wire 1. We can adjust the boundary conditions of wire 2, such that A_2 and B_2 both vanish, then equation (11) fits the pattern of the integral equation of a single wire, from which we can find the equivalent differential equation D_1, such that

$$D_1\mathrm{I}_1^0=\mathrm{I}_1^{0''}+U_1\mathrm{I}_1^{0'}+T_1\mathrm{I}_1^0=0 \tag{13}$$

where, I_1^0 represents the solution on wire 1 having A_2 and B_2 adjusted to zero. We need two solutions of different boundary conditions on wire 1 to determine U_1 and T_1. For robustness, one should use 2 different loadings at the end $-\mathrm{L}_{1a}$ to find U_1 and T_1 between 0 and $-\mathrm{L}_{1a}$ and another 2 different loadings at L_{1b} to find U_1 and T_1 between 0 and L_{1b}.

For a solution I_1 where A_2 and B_2 are not adjusted to zero, we should get,

$$D_1I_1=-A_2f_2(\ell_1)-B_2g_2(\ell_1) \tag{14}$$

where f_2 and g_2 are known functions,

$$f_2(\ell_1)=D_1\boldsymbol{L}_{12}\boldsymbol{L}_{22}^{-1}\sin k\ell_2 \tag{15}$$

$$g_2(\ell_1)=D_1 \boldsymbol{L}_{12}\boldsymbol{L}_{22}^{-1}\cos k\ell_2 \tag{16}$$

The complete differential coupled equation are:

$$I_1''+U_1 I_1'+T_1 I_1+A_2 f_2(\ell_1)+B_2 g_2(\ell_1)=0 \tag{17}$$

$$I_2''+U_2 I_2'+T_2 I_2+A_1 f_1(\ell_2)+B_1 g_1(\ell_2)=\frac{-jV_0}{2Z_0}h_1(\ell_2) \tag{18}$$

where f_1, g_1, f_2, g_2 *and* h_1 are all known functions, and

$I(\ell_1)$, A_1, B_1 *and* $I(\ell_2)$ A_2, B_2 are to be solved in the differential equations.

The moral of the coupling in the integral and differential equations is that the effects of the boundary conditions of one wire on the equations of another wire are the coefficients of two known functions.

Maxwellian Circuits of Multi Wire Systems

Using the differential equation formulation of the coupled integral equations as an ansate to the Maxwellian circuits of a two wire system, we suggest the following form of coupled first order equations for the voltage and currents,

$$\frac{dV_1}{d\ell_1}=-j\omega L_{11}(\ell_1)I_1+\alpha_{11}(\ell_1)V_1-j\omega L_{12}(\ell_1)I_2(L_{2b})+\alpha_{12}(\ell_1)V_2(L_{2b}) \tag{19}$$

$$\frac{dI_1}{d\ell_1}=-j\omega C_{11}(\ell_1)V_1+\beta_{11}(\ell_1)I_1-j\omega C_{12}(\ell_1)V_2(L_{2b})+\beta_{12}(\ell_1)I_2(L_{2b}) \tag{20}$$

$$\frac{dV_2}{d\ell_2}=-j\omega L_{22}(\ell_2)I_2+\alpha_{22}(\ell_2)V_2-j\omega L_{21}(\ell_2)I_1(L_{1b})+\alpha_{21}(\ell_2)V_1(L_{1b}) \tag{21}$$

$$\frac{dI_2}{d\ell_2}=-j\omega C_{22}(\ell_2)V_2+\beta_{22}(\ell_2)I_2-j\omega C_{21}(\ell_2)V_1(L_{1b})+\beta_{21}(\ell_2)I_1(L_{1b}) \tag{22}$$

A total of 4 solutions are needed to find the coefficients on each wire. The robustness of the equations depends on the judicious choices of the solutions, for example, one should not use solutions of varying loads on wire 1 only when we try to find the coefficients of wire 2. Because of the changed dimensions in equations (19) – (22), the coupling coefficients are no longer vector potentials, but voltage and current at fixed points on the other wire. We have chosen the terminal points for convenience. They can be any point on the other wire.

Numerical Results

In Fig. (1) the configuration of 3 coupled lines are shown. A total of 6 solutions of different sources and loads are required to find the circuit parameters. We have chosen 3 different sets of six solutions, the sources and loadings are given in the following table.

Table I Three cases for combinations of excitation voltages and loaded impedances used to examine equation parameter invariance

(Z_{1S}, Z_{2S}, Z_{3S} is the internal impedances of the voltage sources on three lines, respectively)

K=1~6	Z_{1S}	Z_{2S}	Z_{3S}	Z_{1L}	Z_{2L}	Z_{3L}	V_{1S}	V_{2S}	V_{3S}
case 1	j10*k	4*k	200+10*k	j100+50*k	200*k	j10*k	2*k+0.5	k	2*k
case 2	10*k	j10*k	j100+5*k	j50+10*k	j100*k	10*k	k+0.5	3*k	k
case 3	50*k	5*k	10*k	j10*k	j10+k	40+10*k	k	0	0

The parameters all come out the same as shown in Fig.(2). The solutions of the currents of the Maxwellian circuit for a case not one of the above mentioned set are identical to those of the MoM, as shown in Fig. (3).

Conclusion

We have defined the Maxwellian circuit, proved basic theorems and presented computational procedures to obtain the circuit components of Maxwellian circuits for multiple wires. We have corrected the conventional telegrapher's equations by inclusions of the dependent sources. We have shown that the dependent sources are the missing links between Kirchhoff's and Maxwell's formulations. The Maxwellian circuits of multiwires are shown to have mutual capacitances, inductances, and mutual dependent sources, between a point to multipoint, which is quite different from the results we would get should we have used our intuition alone. We have presented numerical results to demonstrate that the theory of Maxwellian circuit extended to multi-wires is still robust.

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Fig. 1 The configuration of a 3-wire system of suspended transmission line.

Fig. 2 Comparisons of Maxwellian circuit parameters obtained from 3 different sets of loadings, (2a) $L_{ij}\ in \mu H/m$, (2b) $C_{ij}\ in \mu f/m$, (2c) α_{ij} and (2d) β_{ij}

Fig. 3 Comparison of current on the wires between results of MoM and Maxwellian circuit of, $V_{1S}=V_{2S}=V_{3S}=1V$, $Z_{1S}=Z_{2S}=Z_{3S}=50\Omega$, $Z_{2L}=Z_{3L}=50\Omega$, $Z_{1L}=10^{10}\Omega$ *(open)*

(3a) real part of current (3b) Imaginary part of current

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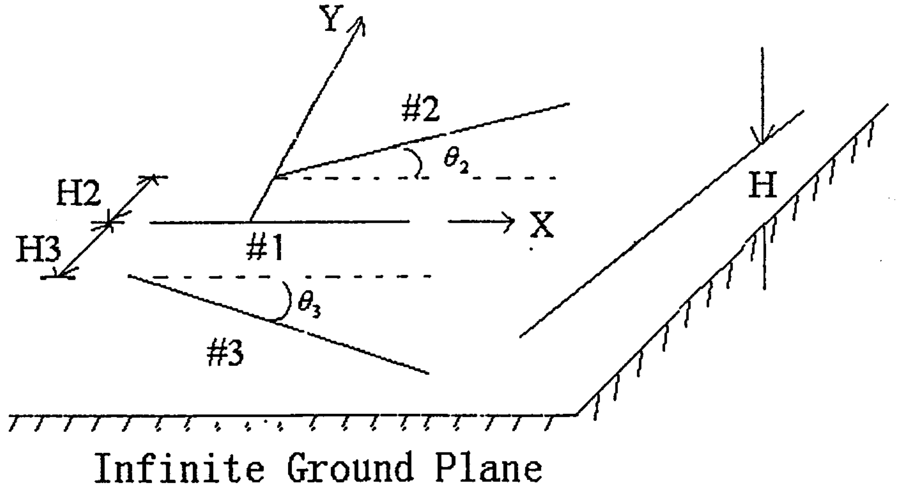

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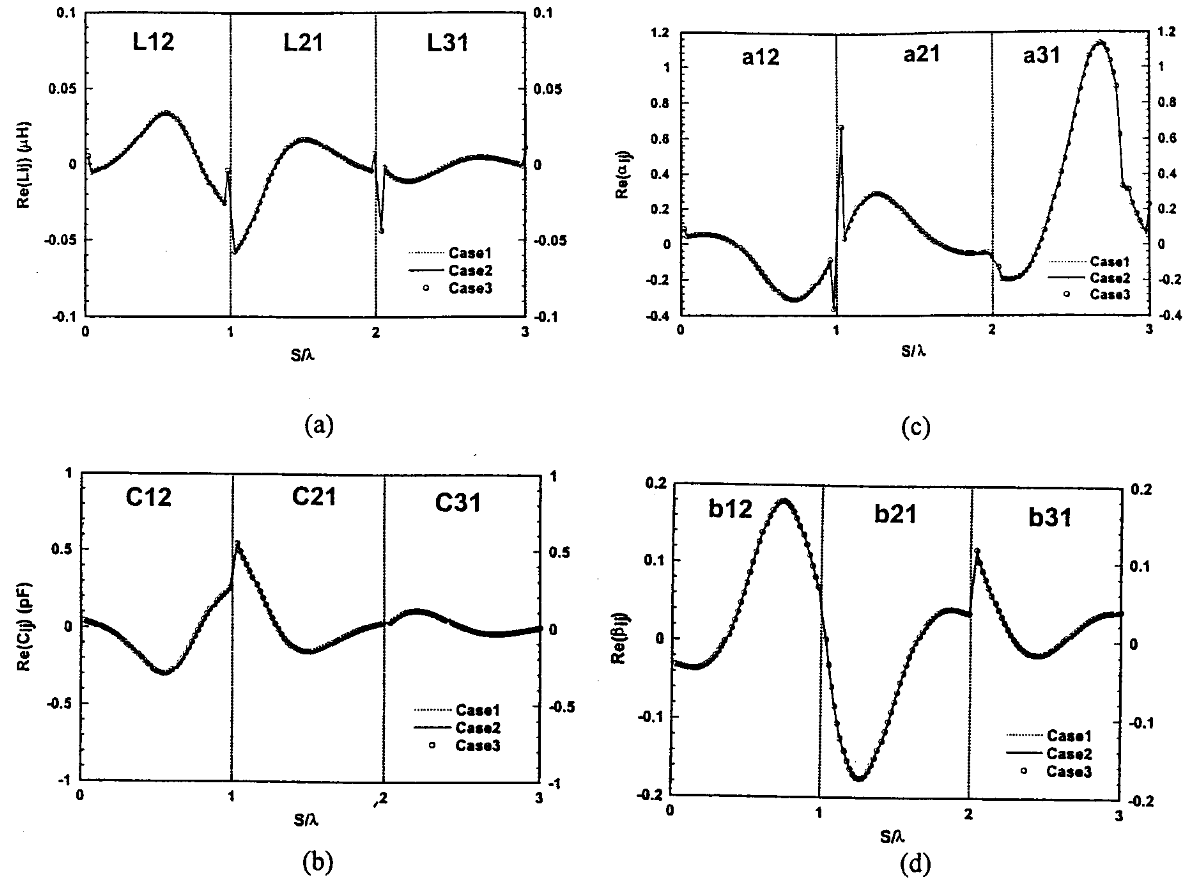

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