EXTRAPOLATION IN TIME AND FREQUENCY DOMAIN DATA USING BESSEL-CHEBYCHEV POLYNOMIALS

Jinhwan Koh, Young-Ki Cho
School of Electrical Engineering and Computer Science
Kyungpook National University
Taegu 702-701, Korea
Email : jikoh@ee.knu.ac.kr, ykcho@ee.knu.ac.kr

Tapan K. Sarkar
Department of Electrical Engineering and Computer Science
Syracuse University
Syracuse, NY, USA
Email : tksarkar@mailbox.syr.edu

1. Introduction

The Method of Moments (MOM) [1], which uses an integral equation formulation, can be used to perform the frequency domain analysis. For broadband analysis, this approach can get very computationally intensive. As the MOM program needs to be executed for each frequency of interest, and for high frequencies, the size of matrix can be very large.

The time domain approach is preferred for broadband analysis. Other advantages of a time domain formulation include easier modeling of non-linear and time-varying media, and use of gating to eliminate unwanted reflections. For a time domain integral equation formulation, the method of Marching on in Time (MOT) [2] is usually employed. A serious drawback of this algorithm is the occurrence of late-time instabilities in the form of high frequency oscillation due to the finite precision of the computation [2].

In this paper we present a technique to overcome late time oscillations. Using early time and low frequency data, we obtain stable late-time and broadband information. The MOM approach can efficiently generate low frequency data, while the MOT algorithm can be used to obtain stable early time data quickly. The overall analysis is thus computationally very efficient.

An optimal choice of basis functions would therefore be one that provides compact support. The Bessel series is well suited for signals with compact support. The fact that Fourier Transform of Bessel functions is an Chebyshev function allows us to work simultaneously with time and frequency domain data. It is better to use the Bessel polynomials instead of Hermite [3] even though they are eigenfunctions of the Fourier transform operator. The problem with Hermite expansion is that these polynomials are two sided (-∞, +∞) and hence the origin of the expansion of the given functions by a Hermite series is very critical. In contrast, one sided Bessel function can be defined over the interval [0, +∞] since the corresponding first kind and second kind of Chebyshev functions form a Hilbert transform pair and hence are considered to be more suited for the problem at hand.

In the next section, we introduce the orthogonal functions and set up the relevant equations for the problem.

2. Bessel-Chebyshev functions

Consider a time sequence $x(t)$, which can be expressed by the orthogonal basis kernel $\phi_n(t,l_1)$.

$$x(t) = \sum_{n=0}^{\infty} a_n \phi_n(t, l_1), \tag{1}$$

where a_n s are the expansion coefficients. One can choose the orthogonal basis function as $\phi_n(t,l_1) = t^{-1} J_n(t,l_1)$ where $J_n(t,l_1)$ is a Bessel function of the first kind of degree *n* and l_1 is a scaling factor.

A signal with compact time support can be expanded as

$$x(t) = \sum_{n=0}^{N} a_n t^{-1} J_n(t, l_1). \tag{2}$$

The Fourier transform of the above expression can be evaluated through [4]

$$X(f) = \sum_{n=0}^{N} a_n \frac{2\ i}{n} (-i)^n \left[1 - \left(\frac{f}{l_2}\right)^2\right]^{1/2} U_{n-1}\left(\frac{f}{l_2}\right) \qquad ;\ |f|^2 < l_2 \tag{3}$$

where $l_2 = 1/2\pi l_1$ and $U_n(f)$ is the Chebyshev polynomial of the second kind defined by

$$U_n(f) = \frac{\sin[(n+1)\cos^{-1} f\,]}{(1-f^2)^{1/2}}. \tag{4}$$

In this expression, the causality in time is not forced whereas the signals we are dealing with are causal. The relationship between the 1st kind of Chebyshev polynomial and 2nd kind of Chebyshev polynomial are the Hilbert transform, i.e.,

$$\int_{-1}^{1} \frac{\sqrt{1-y^2}\,U_{n-1}(y)}{y-x} dy = -\pi T_n(x) \tag{5}$$

Where $T_n(f)$ is the Chebyshev polynomial of the first kind defined by

$$T_n(f) = \cos\left[n \cos^{-1}(f)\right]. \tag{6}$$

Therefore the causal time signal and its Fourier transform can be written as following:

$$x(t) = \sum_{n=0}^{N} a_n t^{-1} J_n(t, l_1) \quad ; t \geq 0$$

$$X(f) = \sum_{n=0}^{N} a_n \frac{2}{n} (-i)^n \left\{ i \left[1 - \left(\frac{f}{l_2}\right)^2\right]^{1/2} U_{n-1}\left(\frac{f}{l_2}\right) + T_n\left(\frac{f}{l_2}\right) \right\} \qquad ;\ |f|^2 < l_2. \tag{7}$$

3. Matrix Formulation

Let M_1 and M_2 be the number of time and frequency domain samples that are given for the functions $x(t)$ and $X(f)$, respectively. Then the matrix representation of time domain data would be,

$$\begin{bmatrix} \phi_0(t_1,l_1) & \cdots & \phi_{N-1}(t_1,l_1) \\ \vdots & \ddots & \vdots \\ \phi_0(t_{M_1},l_1) & \cdots & \phi_{N-1}(t_{M_1},l_1) \end{bmatrix}_{M_1 \times N} \begin{bmatrix} a_0 \\ \vdots \\ a_{N-1} \end{bmatrix}_{N \times 1} = \begin{bmatrix} x(t_1) \\ \vdots \\ x(t_{M_1}) \end{bmatrix}_{M_1 \times 1}, \tag{8}$$

where *N*-1 is the number of the maximum degree in the polynomials.

Similarly in the frequency domain,

$$\begin{bmatrix} \Phi_0(f_1,l_2) & \cdots & \Phi_{N-1}(f_1,l_2) \\ \vdots & \ddots & \vdots \\ \Phi_0(f_{M_2},l_2) & \cdots & \Phi_{N-1}(f_{M_2},l_2) \end{bmatrix}_{M_2 \times N} \begin{bmatrix} a_0 \\ \vdots \\ a_{N-1} \end{bmatrix}_{N \times 1} = \begin{bmatrix} X(f_1) \\ \vdots \\ X(f_{M_2}) \end{bmatrix}_{M_2 \times 1}, \tag{9}$$

where $\phi_n(t,l_1)=t^{-1}J_n\left(\frac{t}{l_1}\right)$, $\Phi_n(f,l_2)=\frac{2i}{n}(-i)^n\left\{i\left[1-\left(\frac{f}{l_2}\right)^2\right]^{1/2}U_{n-1}\left(\frac{f}{l_2}\right)+T_n\left(\frac{f}{l_2}\right)\right\}$.

Combining the two equations we get:

$$\begin{bmatrix}\phi(t,l)\\ \Phi(f,l)\end{bmatrix}_{(M_1+M_2)\times N}\begin{bmatrix}a_0\\ \vdots\\ a_{N-1}\end{bmatrix}_{N\times 1}=\begin{bmatrix}x(t)\\ X(f)\end{bmatrix}_{(M_1+M_2)\times 1}. \qquad (10)$$

N was chosen to make the operator to be a square matrix for utilizing the conjugate gradient method, i.e.,

$$N=M_1+M_2. \qquad (11)$$

7. Numerical Simulations

Consider a helicopter of perfect conductor in the *xy* plane as shown in Fig1. We assumed that the maximum segment size equal to 0.1 times of the wavelength. The maximum segment size in Fig1 is less than 0.5m, which is equivalent to the maximum frequency of 60MHz. The total number of edges of the structure is 2937. The excitation arrives from the direction $\theta = 0$, $\phi = 0$; i.e. along the negative *z* direction. $\vec{u}_i$ is along the *x*-axis. The excitation has the form

$$\vec{E}^{inc}=\vec{u}_i\frac{1}{\sigma\sqrt{\pi}}E_0e^{-\gamma^2}, \quad \text{where } \gamma=\frac{(ct-ct_0-\vec{r}.\vec{k})}{\sigma}, \qquad (12)$$

$\vec{u}_i$ is the unit vector that defines the polarization of the incoming plane wave, E_0 is the amplitude of the incoming wave (377V/m in our example), t_0 is a delay, $\vec{r}$ is the position of an arbitrary point in space, $\vec{k}$ is the unit wave vector defining the direction of arrival of the incident pulse.

Using the MOT algorithm [2], time domain data is obtained from $t = 0$ to $t = 1333.3$*ns* (80 data points). The time step was 16.667n*s* and $\sigma = 83.333$ns. Using MOM program [5], frequency domain data is obtained from DC to $f = 10.5$MHz (70 data points). Assume that only the first 32 time-data points (up to $t = 533.33$*ns*) and first 21 frequency-data points (up to $f = 3.15$MHz) are available. Solving for the matrix equation (10) using the available data, the time domain response is extrapolated to 80 points (1333.3*ns*) and frequency domain response is extrapolated to 70 points (105MHz).

As mentioned earlier, the conjugate gradient method (CGM) has been utilized to solve the matrix equation since it is one of the powerful techniques of solving a matrix equation. It has been carried out until the residue satisfying a certain tolerance value [4]. The tolerance is set to 10^{-6} for this example. The order of expansion was chosen to be 74 (=32+2×21). The expansion coefficients are shown in Fig2. Fig3 is the result in time domain extrapolation and the original have been compared. The vertical lines in all figures indicate where the extrapolations have been performed. It can be seen that the time domain reconstructions are agreeable with the actual (MOT) data. The reconstruction in frequency domain is also reasonably good, as can be seen from Fig4.

All simulations were done in double precision on a PentiumIII 750 PC. Processing time with MOM appraoch takes almost 3 times compared to the processing time using MOT approach. Most of the computation time taken for the proposed approach is to obtain the initial data from MOM and MOT.

8. Conclusions

This paper deals with the problem of simultaneous extrapolation in time and frequency domain using only early time and low frequency data. This has been accomplished by the use of Bessel-Chebyshev polynomials expansions. The computation involved is minimal because we require only early time and low frequency information. In addition, we need to solve a small matrix equation.

In the numerical examples, we have applied this technique to the problem of extrapolating the current on a scatterer being excited by a uniform plane wave. Using early time and low frequency data, we have demonstrated good extrapolation in both time and frequency domain.

References
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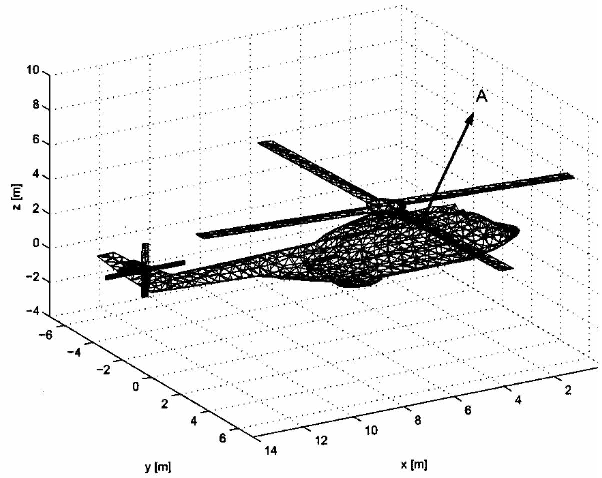

Fig. 1 Discritization of a helicopter

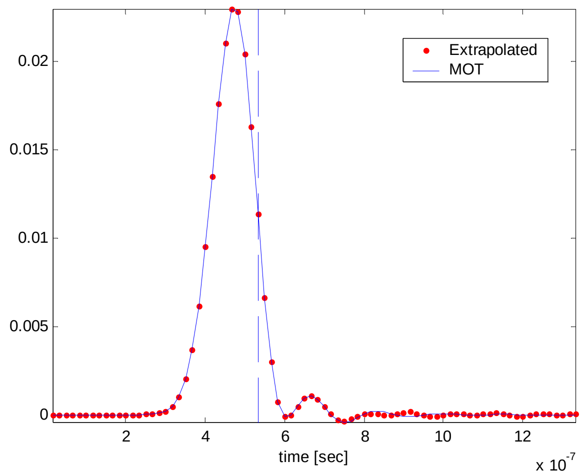

Fig.3 Time domain response

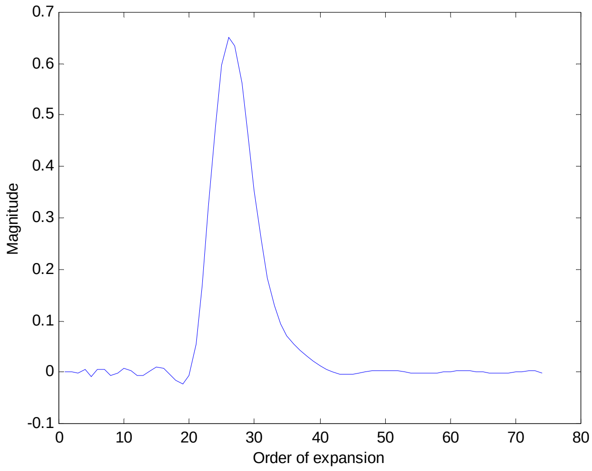

Fig.2 Expansion coefficients

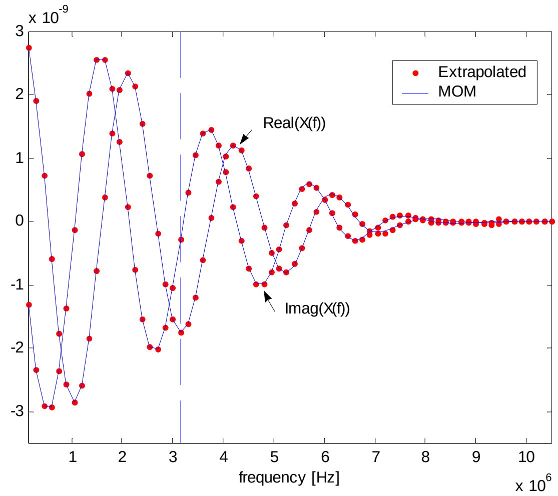

Fig.4 Frequency response