Fast Iterative Algorithm for Solving Matrix Equation in MoM Analysis of Large-scale Array Antennas

Qiang Chen[†], Qiaowei Yuan[††], and Kunio Sawaya[†]
[†]Tohoku University, Sendai, Japan, 980-8579
[††]Oi Electric Co., Ltd., Sendai, Japan, 981-3206

Abstract A new iterative algorithm based on the Gauss-Seidel iteration method is proposed to solve the matrix equation in the MoM analysis of the array antennas. In the new algorithm, the impedance matrix of the array is treated to consist of the self and mutual sub matrices between the neighboring groups of the array, and each sub matrix is regarded as a basic iteration unit rather than the matrix element in the ordinary Gauss-Seidel iteration method. It is found that the convergence condition of ordinary Gauss-Seidel iteration scheme is very strict for the practical use, while the convergence characteristics of the present algorithm are greatly improved. The new algorithm can be applied to the sub domain MoM with a fast convergence if the grouping technique is properly used. The computation time for solving the matrix equation is reduced to be proportional to the square of the number of the array elements, rather than the third power in the Gauss-Jordan method. The present method is effective in MoM analysis of solving lager-scale array antennas.

Key words Antenna, Antenna array, Moment method, Matrix, Iteration

1.Introduction

A large-scale phased array antenna is important and attractive in the rapidly developed mobile communication systems to provide broadband communication with high quality. It is desired to analyze the basic array characteristics numerically, such as the active impedance and active element pattern, at for designing the array antenna. The method of moment (MoM) is one of the efficient methods for the analysis of the array antennas. When the array antenna has N antenna elements and each element is divided into M segments for sub domain MoM analysis, $N_T \times N_T$ matrix equation has to be solved to obtain the unknown current vector, where $N_T = M \times N$.

When the direct method such as the Gauss-Jordan method is employed to solve the matrix equation, the CPU time is proportional to N_T^3. In the case of large-scale array antenna, N becomes so large that the CPU time for solving matrix equation is much longer than that for evaluating the impedance matrix, which is proportional to N_T^2, and becomes the dominant part of the total CPU time in the MoM analysis. The direct method has another problem that the round-off error, which is relatively large, can not be ignored for a large system of equations [1].

Instead of the direct methods, the iterative methods such as the Gauss-Seidel method and the Conjugate Gradient (CG) method have been applied to solve the linear matrix equations. The number of arithmetic operations of these iterative methods is usually proportional to N_T^2 for each iteration step when N_T is very large. However, it was pointed out that the required number of the iteration step of the CG method depends on the analysis model and the size of segments of the basis functions. It is proportional to N_T, which means the total number of arithmetic operation is proportional to N_T^3, the same order to the direct method [3, 4]. Although a fast inhomogeneous plane wave algorithm has been proposed to reduce the operation cost for the matrix multiplication in the CG method to the order of $N_T \log N_T$ [5], it is limited to the 2-D scattering problems. As for the Gauss-Seidel method, the criterion for the convergence of the iteration is that the diagonal values of the impedance matrix are large enough compared to the off-diagonal values [2]. However, the impedance matrix of the MoM analysis does not meet the requirements in most of cases, and the Gauss-Seidel iterative method does not converge. Therefore, it is required to develop a fast and stable iterative algorithm whose computation cost is less than $\mathrm{O}(N^3)$ for solving the matrix equation in order to perform the MoM analysis of a large-scale array antenna.

In this paper, a new iterative algorithm based on the Gauss-Seidel method is proposed to solve the matrix equation $[Z][I]=[V]$ in the MoM analysis of the array antennas, whose CPU time is approximately $\mathrm{O}(N^2)$. The convergence criterion of the iterative algorithm is investigated and the effect of the method is shown by some numerical examples.

2. GAUSS-SEIDEL SCHEME

The important procedure for solving the matrix equation $[Z][I]=[V]$ for unknown $[I]$ by using the Gauss-Seidel scheme is to split the matrix $[Z]$ into $[Z]=[S]+[T]$ so that the matrix equation becomes

$$[S][I]=-[T][I]+[V], \qquad (1)$$

where $[S]$ has the lower-left triangular part including the diagonal elements of $[Z]$, and $[T]$ has the upper-right triangular part excluding the diagonal elements. The iterative scheme for solving Eq. (1) is given by:

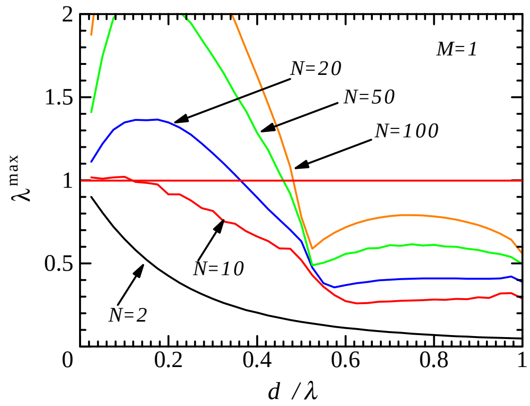
$$I_i^{(l+1)}=\frac{1}{S_{ii}}\left(V_i-\left(\sum_{j=1}^{i-1}S_{ij}I_j^{(l+1)}+\sum_{j=i+1}^{MN}T_{ij}I_j^{(l)}\right)\right) \qquad (2)$$
$$i=1,2,\cdots,K;\; l=1,2,\cdots,L.$$

where I_i, S_{ij} and T_{ij} are the elements of the vector $[I]$, matrices $[S]$ and $[T]$, respectively. The superscript l is the step number of the iteration. The initial $I_i^{(0)}$ is usually assumed to be zero. This iteration continues until $|I_i^{(l+1)}-I_i^{(l)}|\le\varepsilon$ for all i at the final Lth step. The convergence criterion for the Gauss-Seidel scheme is that all the eigenvalues of the matrix $[S]^{-1}[T]$ have their magnitudes less than unity [1].

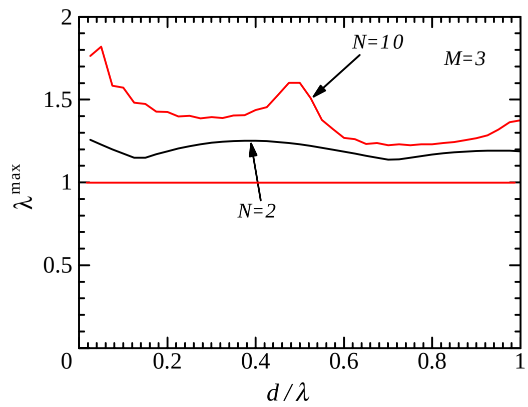
Fig. 1. Analysis model: a linear dipole array antenna.

Piece-wise sinusoidal (PWS) MoM analyses [2] is performed to show whether the iteration method can be applied or not. The analysis model of the linear dipole antenna array is shown Fig. 1. Each dipole element is divided into M overlapped dipole segments.

Fig. 2 shows the largest magnitude of the eigenvalues of $[S]^{-1}[T]$ obtained from the PWS MoM analysis for a half wavelength dipole array when M is 1 and 3. Fig. 2(a) shows the case of M=1, which means each dipole element is not divided into segments. The largest magnitude $\lambda^{\max}$ of the eigenvalues is smaller than unity for two cases, one is that d is larger than λ/2, and the other case is that d is smaller than λ/2 but N is limited to relatively small number, where λ is the wavelength. In both cases, the array elements are divided into the dipole segments. It is also indicated that the eigenvalues become large as the total number of the array elements N increases. Fig. 2(b) shows that $\lambda^{\max}$ is always larger than unity when M is equal to 3.



(a) M =1



(b) M=3

Fig. 2. The largest magnitude of eigenvalues of matrix $[S]^{-1}[T]$ versus array spacing d for PWS MoM.

The above numerical results show that the Gauss-Seidel method can not be applied to solve the matrix equation in MoM for the usual array antennas. The convergence criterion of the convergence depends on the total number of the array elements, the number of the segments for each element, the array spacing, and the geometry of the antenna. Therefore, it is necessary to improve the convergence characteristics of the iteration.

3. NOVEL ITERATIVE ALGORITHM

In order to overcome the difficulty mentioned above, a novel iterative algorithm is proposed. The iterative unit is changed to the sub matrices that include the self and mutual impedance between the neighboring groups of the array, and the sub matrices the basic iteration units rather than the matrix element in the ordinary Gauss-Seidel iteration method. If each group consists of K elements, and the total array elements are divided into N/K groups completely as shown in Fig. 3, the iterating procedure is expressed by:

$$[\bar{I}]_i^{(l+1)} = [\bar{I}]_i^{(0)} - [\bar{Z}]_{ii}^{-1}\left[\sum_{j=1}^{i-1}[\bar{Z}]_{ij}[\bar{I}]_j^{(l+1)} + \sum_{j=i+1}^{N}[\bar{Z}]_{ij}[\bar{I}]_j^{(l)}\right]^{T}, \quad (3)$$

$$i = 1, 2, \cdots, N/K; \; l = 1, 2, \cdots, L.$$

where $[\bar{I}]_j$ is a MK current vector of the group i, and $[\bar{Z}]_{ii}$ is a $MK \times MK$ matrix, which means the self and mutual impedance of the dipole segments between two groups i and j. For fast convergence, the initial $[\bar{I}]_i^{(0)}$ is assumed to the current on a single group of the array elements ignoring the mutual coupling between the groups, which is given by

$$[\bar{I}]_i^{(0)} = [\bar{Z}]_{ii}^{-1}[\bar{V}]_i, \qquad i = 1, 2, \cdots, N/K. \quad (4)$$

where $[\bar{V}]_i$ is the voltage vector of group i.

The convergence criterion for this algorithm is investigated numerically by analyzing the same model shown in Fig. 3. First, the convergence characteristics are investigated in the case of K=1, which means the array is not divided into groups.

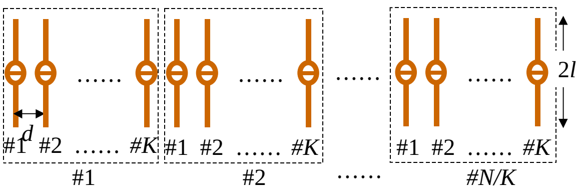


Fig. 3. Analysis model for the novel iterative algorithm: the linear dipole array antenna is divided into N/K groups.

Fig. 4 shows the convergence criterion of the array spacing d versus the number of the array elements N when the dipole length is λ/2. Although Fig. 4 shows the case of M=3, it is found the convergence characteristics depend on the number of the array elements N and the array spacing d, but are almost independent of the segment number M of each array element. The figure illustrates that the novel iteration algorithm converges when d is larger than λ/2, or when d is smaller than λ/2 but N is limited to a relatively small number, which is similar to the case of M=1 in the Gauss-Seidel method even though the array element is divided into several segments in the present analysis.

Although the convergence criterion of present method is improved compared with that of the original Gauss-Seidel method, the divergence area still remains for K=1. Therefore, the convergence characteristics is examined for the case of K>1.

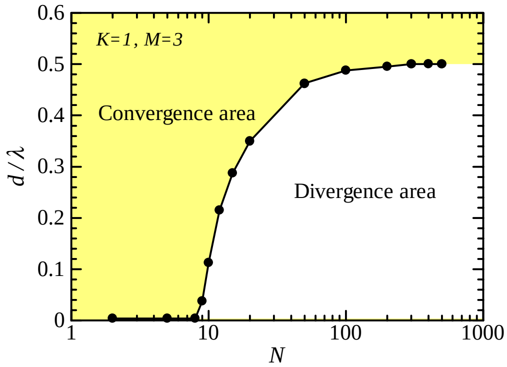


Fig. 4. Convergence criterion of array spacing d for various array element number N, when dipole length is λ/2.

Fig. 5 shows the iteration steps required for Eq. (3) when M=9 and N=100. The curve of K=1, which means the grouping technique is not applied, is shown only in the range of 0.5 to 1 because the iteration diverges when d/λ is smaller than 0.5 as shown in Fig. 4. However, it is found that if the value of K increases to over 10, the iteration converges even when d/λ is as small as 0.04. Therefore, the grouping technique makes the iteration much more stable so that the iterative criterion is improved. When K increases, the curve of the required iteration steps becomes more and more flat, which means the required number of the iteration steps becomes independent of the array spacing.

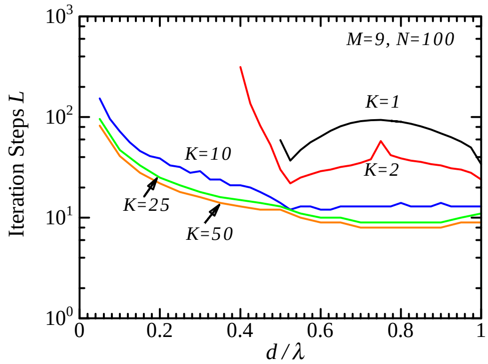


Fig. 5. Iteration steps required to perform Eq. 5 when every K elements are grouped, M=9 and N=100.

Although a large K can reduce the iteration steps, too large K would result in consuming a long CPU time. The total computational time T can be estimated by the expression

$$T = \alpha(KM)^3 + \beta L(MN/K)^2, \quad (5)$$

where the first term is for evaluating $[Z]_{ii}^{-1}$, the second term is for iterating process, and the α, β are constants depending on the computer performance. Therefore, a large value of K can improve the convergence characteristics and decrease the required number of the iteration steps, but increase the CPU time for evaluating $[Z]_{ii}^{-1}$ which is proportion to the third power of K.

The number of iteration steps versus the total number of the dipole array elements is shown in Fig. 6 for the case of $2l=\lambda/2$ and $d=\lambda/2$. When K is large, the number of iteration steps becomes independent of the element number N. If the number of iteration steps is independent of the element number N, the computational cost consumed by the present method would be approximately proportional to N^2 when N is as large as the value of second term shown in Eq. (5) is much greater than the value of the first term. The CPU time versus N is shown in Fig. 7. The value of the CPU time shown was measured by using a Pentium-III 450MHz PC with a 256 MB memory. The curve of the Gauss-Jordan method is also plotted for comparison. As expected the CPU time is proportional to N^3 by using the Gauss-Jordan method, while to N^2 by using the present method with a proper K. The cost saving effect of the numerical computation is significant.

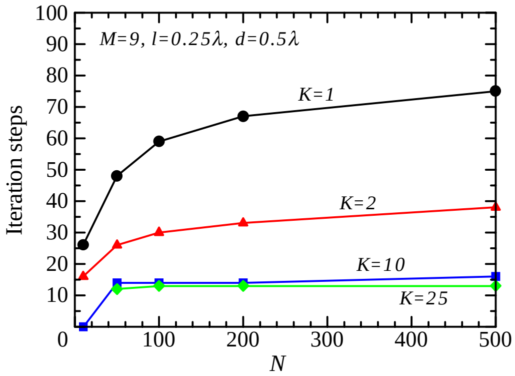


Fig. 6. Iteration steps versus total number of array elements with various K.

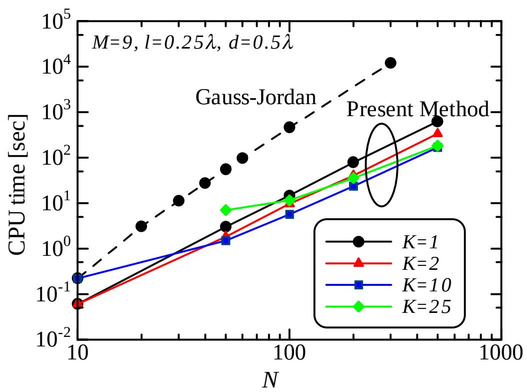


Fig. 7. CPU time versus total number of array elements with various K.

4. CONCLUSION

An iterative algorithm has been proposed to solve the matrix equation of the MoM analysis for the array antenna. The convergence criterion of the iterative algorithm has been investigated numerically. The CPU time has been shown to be approximately proportional to N^2, which is expected to apply the MoM analysis to the large-scale array antennas.

It should be noted that the Gauss-Seidel method, which is applied to the present method, is not the most ideal to solve the linear matrix equation. Some improving techniques such as the Back and Forth Seidel Process [7], the method of Successive Overrelaxation (SOR) [8] are superior to the original Gauss-Seidel method on the aspects of stability and convergence. However these techniques can also be directly applied to the present method. This is future work of this study.

References

[1] Tapan K. Sarkar, Kenneth R. Siarkiewicz, and Roy F Straton, "Survey of Numerical Methods for solution of Large Systems of Linear Equations for Electromagnetic Field Problems," *IEEE Trans. Antennas Propagat.*, vol. AP-29, pp. 847-856, Nov. 1981.

[2] Myron B. Allen III and Eli L. Isaacson, "NUMERICAL ANALYSIS FOR APPLIED SCIENCE," John Wiley & Sons, Inc. 1998.

[3] Andrew F. Peterson and Raj Mittra, "Convergence of the Conjugate Gradient Method when Applied to Matrix Equations Representing Electromagnetic Scattering Problems," *IEEE Trans. Antennas Propagat.*, Vol. AP-34, pp. 1447-1454, Dec. 1986.

[4] A. F. Peterson, C. F. Smith and R. Mittra, "Eigenvalues of Moment-Method Matrix and Their Effect on Conjugate Gradient Algorithm," *IEEE Trans. Antennas Propagat.*, Vol. AP-36, pp. 1177-1179, Aug. 1988.

[5] Bin Hu, Weng Cho Chew, Eric Michielessen and Junsheng Zhao, "Fast inhomogeneous plane wave algrithm for the fast analysis of two-dimensional scattering problems," *Radio Science*, vol. 34, pp. 759-772, July-Aug. 1999.

[6] Jack. H. Richmond and N. H. Greay, "Mutual impedance of non-planar-skew sinusoidal dipoles, " *IEEE Trans. Antennas and Propagat.*, vol.AP-23, pp.412-414, May 1975.

[7] G. E. Forsythe, "Solving linear equation can be interesting," *Bull. of Am. Math, Society*, pp. 299-329, 1953.

[8] D. Yong, "On the solution of large system of linear algebraic equations with sparse positive definite matrices," in *Numerical Solution of Nonlinear Algebraic Equations*, New York: Academic, 1974.