COMPUTATION OF GENERALIZED CONSTITUTIVE RELATIONS FOR METAMATERIALS

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## 1 Introduction

In recent years, there has been an increasing interest in the development of new materials with characteristics which may not be found in nature. Examples are meta-materials [1][2], composite medium, left-handed medium [3][4], and chiral medium. They have a broad range of applications including artificial dielectric, lens, absorbers, antenna structures, optical and microwave components, frequency selective surfaces, and composite material. In addition, the left-handed material has a negative refractive index, and its permeability and permittivity are both negative, and appear to be capable of producing a perfect lens [5], though its limitations have been pointed out [6].

In these applications, it is important to describe the material characteristics in terms of the physical properties of the inclusions. This paper presents a generalized matrix representation of the macroscopic constitutive relations based on the quasi-static Lorentz-Lorenz theory. The constitutive relations are given by

$$\begin{bmatrix} \bar{D} \\ \bar{B} \end{bmatrix} = \begin{bmatrix} \bar{\bar{\epsilon}} & \bar{\bar{\xi}} \\ \bar{\bar{\zeta}} & \bar{\bar{\mu}} \end{bmatrix} \begin{bmatrix} \bar{E} \\ \bar{H} \end{bmatrix} \tag{1}$$

where $\bar{\bar{\epsilon}}$, $\bar{\bar{\xi}}$, $\bar{\bar{\zeta}}$, and $\bar{\bar{\mu}}$ are 3×3 matrices.

The medium with the constitutive relation (1) is called the "bi-anisotropic medium" [7]. If $\bar{\bar{\epsilon}}$, $\bar{\bar{\xi}}$, $\bar{\bar{\zeta}}$, and $\bar{\bar{\mu}}$ are scalars, this is called the "bi-isotropic" or "chiral" medium. There are two important questions. First is how to obtain these parameters $\bar{\bar{\epsilon}}$, $\bar{\bar{\xi}}$, $\bar{\bar{\zeta}}$, and $\bar{\bar{\mu}}$ for a given material. The second is how to describe the wave characteristics in such a medium.

In this paper, we address the first question. We derive the explicit expressions of these matrix parameters for a given configuration of the inclusions in a host material. The inclusions are arranged in a three-dimensional array and consist of non-magnetic materials with complex dielectric constants. The derivation is based on the quasi-static Lorentz-Lorenz theory and, therefore, applicable to inclusions whose sizes and spacings are small compared with a wavelength.

## 2 Formulation of the Problem

Let us consider a medium consisting of a three-dimensional periodic array of inclusions in a host material (Figure 1). Each inclusion may be a wire, a ring, a helix, or a split ring proposed by Pendry et al [1]. The spacings along $x$, $y$, and $z$ directions are $a$, $b$, and $c$ respectively. Under the influence of applied electromagnetic field, the inclusion produces electric and magnetic multipoles. In this paper, we limit ourselves to the electric and magnetic dipoles expressed in a quasi-static approximation.

The electric and magnetic dipole moments, $\bar{p}_e$ and $\bar{p}_m$, of each inclusion is produced by the local field $[\bar{E}_\ell, \bar{H}_\ell]$ which is the field due to all other dipoles surrounding this particular inclusion. The dipole moments are then given by the generalized polarizability matrix $[\bar{\bar{\alpha}}]$.

$$\begin{bmatrix} \bar{p}_e \\ \bar{p}_m \end{bmatrix} = [\bar{\bar{\alpha}}] \begin{bmatrix} \bar{E}_\ell \\ \bar{H}_\ell \end{bmatrix} = \begin{bmatrix} \bar{\bar{\alpha}}_{ee} & \bar{\bar{\alpha}}_{em} \\ \bar{\bar{\alpha}}_{me} & \bar{\bar{\alpha}}_{mm} \end{bmatrix} \begin{bmatrix} \bar{E}_\ell \\ \bar{H}_\ell \end{bmatrix}. \tag{2}$$

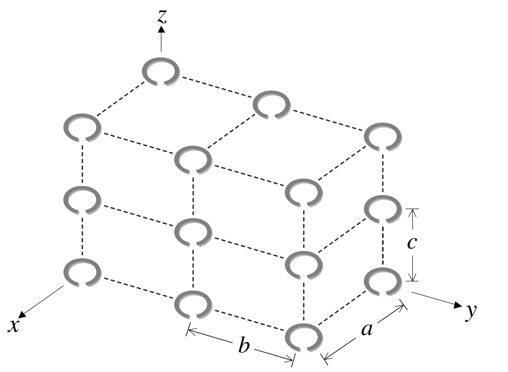


Figure 1: Three-dimensional array of inclusions whose dielectric constant is $\epsilon_r$ and conductivity is $\sigma$. Host material has the dielectric constant $\epsilon_b$.

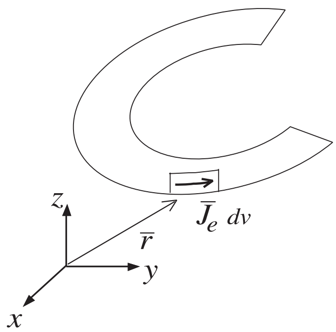


Figure 2: Current $\bar{J}_e$ on the inclusion produced by the local field $\bar{E}_\ell$. Current $\bar{J}_m$ is also produced by the local field $\bar{H}_\ell$.

Note that $\bar{p}_e$, $\bar{p}_m$, $\bar{E}_\ell$, and $\bar{H}_\ell$ are all 3×1 vectors and $[\bar{\bar{\alpha}}]$ is a 6×6 matrix.
The local field $[\bar{E}_\ell,\, \bar{H}_\ell]$ are incident upon the inclusion and produce the current $[\bar{J}_e,\, \bar{J}_m]$, which in turn produce the electric dipole moment $\bar{p}_e$ and the magnetic dipole moment $\bar{p}_m$ given by (Figure 2).

$$
\begin{aligned}
[\bar{p}_e] &= \frac{1}{j\omega} \int dv \ \left( [\bar{J}_e][\bar{E}_\ell] \ + \ [\bar{J}_m][\bar{H}_\ell] \right), && (3) \\
[\bar{p}_m] &= \frac{\mu_0}{2} \int dv \ \bar{r} \times \left( [\bar{J}_e][\bar{E}_\ell] \ + \ [\bar{J}_m][\bar{H}_\ell] \right). && (4)
\end{aligned}
$$

Now we have the final expressions for all the components of the polarizability matrix $[\bar{\bar{\alpha}}]$.

$$
\begin{aligned}
\bar{\bar{\alpha}}_{ee} &= \frac{1}{j\omega} \int dv \ [\bar{J}_e] & \bar{\bar{\alpha}}_{em} &= \frac{1}{j\omega} \int dv \ [\bar{J}_m] \\
\bar{\bar{\alpha}}_{me} &= \frac{\mu_0}{2} \int dv \ \bar{r} \times [\bar{J}_e] & \bar{\bar{\alpha}}_{mm} &= \frac{\mu_0}{2} \int dv \ \bar{r} \times [\bar{J}_m].
\end{aligned} \tag{5}
$$

Let us next consider the local field $[\bar{E}_\ell,\, \bar{H}_\ell]$. As discussed above, the local field acts on the inclusion and produces the electric and the magnetic dipoles. The local field consists of the applied field $[\bar{E},\, \bar{H}]$ and the interaction field $[\bar{E}_i,\, \bar{H}_i]$.

$$
\begin{bmatrix} \bar{E}_\ell \\ \bar{H}_\ell \end{bmatrix} = \begin{bmatrix} \bar{E} \\ \bar{H} \end{bmatrix} + \begin{bmatrix} \bar{E}_i \\ \bar{H}_i \end{bmatrix} \tag{6}
$$

The interaction field is produced by all the dipoles except the particular inclusion under consideration. For a three-dimensional array of inclusions, the interaction field $[\bar{E}_i,\, \bar{H}_i]$ has been obtained and is given by [7]

$$
\begin{bmatrix} \bar{E}_i \\ \bar{H}_i \end{bmatrix} = \mathrm{N}[\bar{\bar{\mathrm{C}}}] \begin{bmatrix} \bar{p}_e \\ \bar{p}_m \end{bmatrix} = \mathrm{N} \begin{bmatrix} \dfrac{1}{\epsilon_0 \epsilon_b} \bar{\mathrm{C}} & \bar{0} \\ \bar{0} & \dfrac{1}{\mu_0} \bar{\mathrm{C}} \end{bmatrix} \begin{bmatrix} \bar{p}_e \\ \bar{p}_m \end{bmatrix} \tag{7}
$$

where N is the number of inclusions per unit volume and is equal to $\mathrm{N} = (abc)^{-1}$, $\bar{0} = 3{\times}3$ null matrix, $\epsilon_b$ is the relative dielectric constant of the host material, and $\bar{\mathrm{C}}$ is the interaction constant matrix $= \begin{bmatrix} \mathrm{C}_x & 0 & 0 \\ 0 & \mathrm{C}_y & 0 \\ 0 & 0 & \mathrm{C}_z \end{bmatrix}$. The interaction constant matrix $[\bar{\mathrm{C}}]$ for a three-dimensional

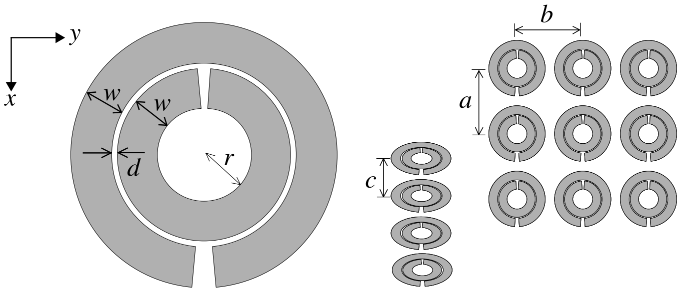


Figure 3: Split ring resonator (SRR) and definitions of distances.

array of dipoles with the spacings $a$, $b$, and $c$ in the $x$, $y$, and $z$ directions respectively (Figure. 1), has been obtained [8].

$$\mathrm{C}_x = f\left(\frac{b}{a},\frac{c}{a}\right) = \left(\frac{b}{a}\right)\left(\frac{c}{a}\right)\left[\frac{1.202}{\pi} - S\left(\frac{b}{a},\frac{c}{a}\right)\right],\ \mathrm{C}_y = f\left(\frac{c}{b},\frac{a}{b}\right),\ \mathrm{C}_z = f\left(\frac{a}{c},\frac{b}{c}\right) \tag{8}$$

where

$$S\left(\frac{b}{a},\frac{c}{a}\right) = \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{m=1}^{\infty}(2m\pi)^2\ \mathrm{K}_0\left(2m\pi\left[\left(\frac{nb}{a}\right)^2+\left(\frac{sc}{a}\right)^2\right]^{1/2}\right),$$

$\mathrm{K}_0$ is the modified Bessel function, and the term with $n = s = 0$ is excluded.
Substituting (2) and (7) into (6), we obtain

$$\begin{bmatrix}\bar{E}_\ell\\ \bar{H}_\ell\end{bmatrix} = \begin{bmatrix}\bar{E}\\ \bar{H}\end{bmatrix} + \mathrm{N}[\bar{\bar{\mathrm{C}}}][\bar{\bar{\alpha}}]\begin{bmatrix}\bar{E}_\ell\\ \bar{H}_\ell\end{bmatrix} = \left[\bar{\bar{\mathrm{U}}} - \mathrm{N}[\bar{\bar{\mathrm{C}}}][\bar{\bar{\alpha}}]\right]^{-1}\begin{bmatrix}\bar{E}\\ \bar{H}\end{bmatrix} \tag{9}$$

where $[\bar{\bar{\mathrm{U}}}]$ is 6×6 unit matrix. Finally, $\bar{D}$ and $\bar{B}$ are given by

$$\begin{bmatrix}\bar{D}\\ \bar{B}\end{bmatrix} = \begin{bmatrix}\epsilon_0\epsilon_b\bar{E}\\ \mu_0\bar{H}\end{bmatrix} + \mathrm{N}\begin{bmatrix}\bar{p}_e\\ \bar{p}_m\end{bmatrix}. \tag{10}$$

Substituting (2) and (9) into (10), we get

$$\begin{bmatrix}\bar{D}\\ \bar{B}\end{bmatrix} = \begin{bmatrix}\bar{\bar{\epsilon}} & \bar{\bar{\xi}}\\ \bar{\bar{\zeta}} & \bar{\bar{\mu}}\end{bmatrix}\begin{bmatrix}\bar{E}\\ \bar{H}\end{bmatrix}$$

where

$$\begin{bmatrix}\bar{\bar{\epsilon}} & \bar{\bar{\xi}}\\ \bar{\bar{\zeta}} & \bar{\bar{\mu}}\end{bmatrix} = \begin{bmatrix}\epsilon_0\epsilon_b\bar{\mathrm{U}} & \bar{0}\\ \bar{0} & \mu_0\bar{\mathrm{U}}\end{bmatrix} + \mathrm{N}[\bar{\bar{\alpha}}]\left[\bar{\bar{\mathrm{U}}} - \mathrm{N}[\bar{\bar{\mathrm{C}}}][\bar{\bar{\alpha}}]\right]^{-1}, \tag{11}$$

$[\bar{\mathrm{U}}]$ is 3×3 unit matrix.
This is the final expression for the generalized constitutive relations for a three-dimensional array of inclusions under quasi-static approximation. This is the generalization of Lorentz-Lorenz formula and leads to the Maxwell-Garnett formula for spherical inclusions.

## 3 Split Ring Resonator

As an example, we consider a three-dimensional array of the Pendry's split ring resonators (SRR) (Figure 3). In this example, $\bar{H}_\ell = H_{\ell z}\hat{z}$ and $\bar{E}_\ell = E_{\ell x}\hat{x} + E_{\ell y}\hat{y}$. Therefore, writing $\bar{\bar{\epsilon}} = [\epsilon_{ij}]$, $\bar{\bar{\xi}} = [\xi_{ij}]$, $\bar{\bar{\zeta}} = [\zeta_{ij}]$, and $\bar{\bar{\mu}} = [\mu_{ij}]$ with $i$ and $j = x$,$y$,$z$, we calculated (Figure 4):

$$\begin{aligned}\epsilon_{xx} &= \epsilon'_{xx} - j\epsilon''_{xx} \qquad \mu_{zz} = \mu'_{zz} - j\mu''_{zz}\\ \epsilon_{yy} &= \epsilon'_{yy} - j\epsilon''_{yy}\end{aligned} \tag{12}$$

All other elements are negligibly small. It is interesting to note the resonance behaviors discussed by Pendry, Smith and others. Also noted are the negative permeability and permittivity which have strong dispersive characteristics as already discussed by several workers.

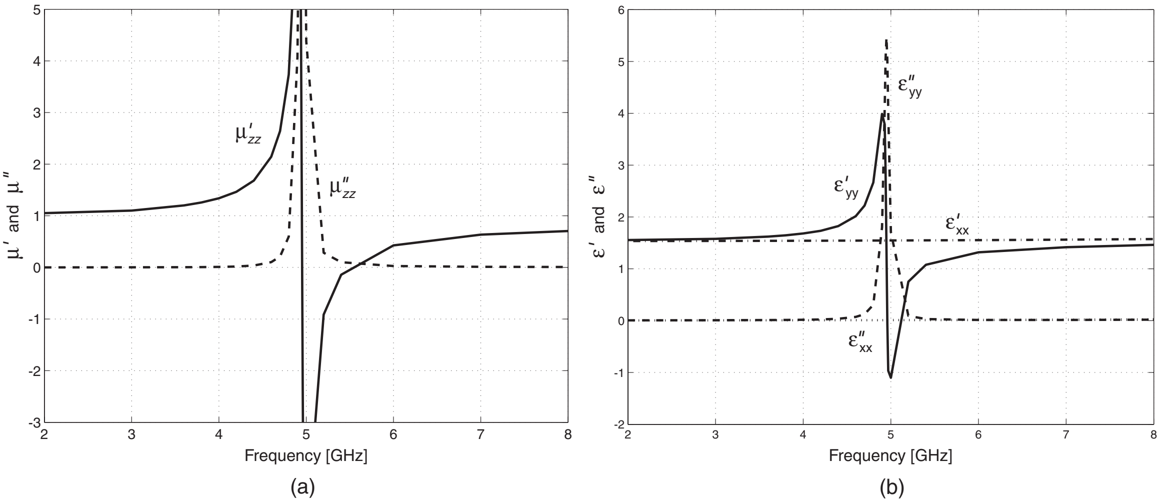


Figure 4: Plot of $\mu$ and $\epsilon$ of a SRR medium. $r = 1.5$mm, $w = 0.8$mm, $d = 0.2$mm, $a = b =$ 8mm, and $c = 3.2$mm. The conductivity of the ring is $5.8 \times 10^7$[S/m]. The thickness of the ring is much greater than the skin depth.

## 4 Conclusions

In the paper, we derived the generalized constitutive relations (11) for meta-materials consisting of a three-dimensional array of inclusion of arbitrary shape. This formula is derived under quasi-static approximation and is applicable to the spacing between the inclusions, which are small compared with a wavelength. Some numerical examples are shown to illustrate the usefulness of the formulation.

## Acknowledgments

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