A-027

Sufficient Condition for Open Rectangle-of-Influence Drawings of Inner Triangulated Plane Graphs*

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Abstract

A straight-line drawing of a plane graph is called an open rectangle-of-influence drawing if there is no vertex in the proper inside of the axis-parallel rectangle defined by the two ends of every edge. In an inner triangulated plane graph, every inner face is a triangle although the outer face is not necessarily a triangle. A sufficient condition for an inner triangulated plane graph G to have an open rectangle-of-influence drawing was known in [4]. In this paper, we improve the condition in [4] and present a linear time algorithm to construct an open rectangle-of-influence drawing of G if G satisfies our condition.

1 Introduction

Recently automatic aesthetic drawing of graphs has created intense interest due to their broad applications, and as a consequence, a number of drawing methods have come out [2, 5]. The most typical drawing of a plane graph G is a *straight-line drawing*, in which all vertices of G are drawn as points and all edges are drawn as straight-line segments without any edge-intersection. A straight-line drawing is called a *grid drawing* if all vertices are put on grid points of integer coordinates.

There are many results on grid drawings under additional constraints. In the paper, we deal with a type of grid drawing, known as the "rectangle-of-influence drawing" of a plane graph[1, 3]. A *rectangle-of-influence* of an edge e is an axis-parallel rectangle having e as one of its diagonals. In each of Figs. 1(a)–(c) a rectangle-of-influence is shaded for an edge $e = (u, v)$ drawn by a thick line. We call a grid drawing a *rectangle-of-influence drawing* (or simply an *RI-drawing*) if there is no vertex in a rectangle-of-influence of any edge. Figures 1(a) and (b) depict RI-drawings, while Fig. 1(c) depicts a grid drawing which is not an RI-drawing. An RI-drawing often looks pretty, since vertices tend to be separated from edges. A rectangle-of-influence of an edge e is *closed* if it contains the boundary of a rectangle, and is *open* if it does not contain the boundary. In a *closed RI-drawing* every rectangle-of-influence is regarded as a closed one, while in an *open RI-drawing* every rectangle-of-influence is regarded as an open one. In a closed RI-drawing, there is no vertex except the ends not only in the proper inside of a rectangle-of-influence of each edge but also on the boundary, as illustrated in Fig. 1(a). In an open RI-drawing, there may be a vertex other than the ends on the boundary of a rectangle, as illustrated in Fig. 1(b). Thus a closed RI-drawing is an open RI-drawing, but an open RI-drawing is not always a closed RI-drawing.

Biedl *et al.* [1] showed that a plane graph G has a closed RI-drawing if and only if G has no filled 3-cycle, that is, a cycle of three vertices such that there is a vertex in the proper inside, and showed that G has a closed RI-drawing on an $(n-1) \times (n-1)$ grid if G has no filled 3-cycle, as illustrated in Fig. 1(a), where n is the number of vertices in G. They also showed that one can test in linear time whether a given planar graph has a plane embedding without filled 3-cycles. A plane graph G may have an open RI-drawing even if G has a filled 3-cycle.

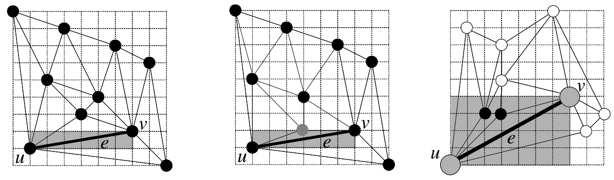

Figure 1: (a) A closed RI-drawing, (b) an open RI-drawing, and (c) a non-RI-drawing of an inner triangulated plane graph without filled 3-cycles.

Miura *et al.* [4] gave a sufficient condition for an inner triangulated plane graph G to have an open rectangle-of-influence drawing. Their condition is expressed in terms of a labeling of angles of a subgraph G^* of G with integers 0, 1, 2, 3 and 4. The subgraph G^*, called the *frame graph* of G, is obtained from G by removing all vertices and edges in the proper inside of every maximal filled 3-cycle of G. G^* is an inner triangulated plane graph and has no filled 3-cycle, and hence G^* has a closed RI-drawing. They showed that G has an open RI-drawing if G^* has a kind of an angular labeling, called a "good labeling." They also present an $O(n^{1.5}/\log n)$-time algorithm to examine whether G^* has a good labeling and, if so, construct an open RI-drawing of G on an $(n-1) \times (n-1)$ grid from a good labeling of G^*. However, a necessary and sufficient condition for an open RI-drawing has not been known.

In this paper, we improve Miura *et al.*'s condition. We show that an inner triangulated plane graph G has an open RI-drawing if G^* has a kind of an angular labeling. We also present a liner-time algorithm to construct an open RI-drawing of G on an $(n-1) \times (n-1)$ grid if G^* has such a labeling.

2 Preliminaries

In this section, we present some definitions and a known result.

We deal with an undirected simple graph G. The set of vertices in G is denoted by $V(G)$. An edge joining vertices u and v is denoted by (u, v). Throughout the paper we denote by n the number of vertices of a graph G. A $W \times H$ *integer grid* consists of $W + 1$ vertical grid lines and $H + 1$ horizontal grid lines. We call W and H the *width* and *height* of the integer grid, respectively.

A graph is *planar* if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A *plane graph* is a planar graph with a fixed embedding. A plane graph G divides the plane into connected regions, called *faces*. The boundary of a face is called a *facial cycle*, and is denoted by a sequence of the vertices on the boundary. The boundary of the outer face is called the *outer facial cycle* of G. A vertex on the outer facial cycle is called an *outer vertex*, while a vertex not on the outer facial cycle is called an

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inner vertex. An edge on the outer facial cycle is called an *outer edge*, while an edge not on the outer facial cycle is called an *inner edge*. A cycle of three vertices in G is called a *3-cycle*. A 3-cycle of G is *filled* if there is a vertex in the proper inside of C. A plane graph G is *inner triangulated* if G is 2-connected and every inner facial cycle is a 3-cycle.

An *angle* of a frame graph G^* is an angle of (a straight-line drawing of) a face of G^*. An angle of an inner face of G^* is called an *inner angle*, while an angle of the outer face is called an *outer angle*. At each vertex v in G^*, draw two lines, one with slope 0 and the other with slope ∞. These two lines define four half-lines at v. We say that an angle at v contains a number i of the four half-lines, $0 \leq i \leq 4$, if the region of the plane defined by that angle contains i half-lines at v. (Note that there is no horizontal or vertical edge in G^*.) A labeling is called a "*good labeling*" of a frame graph G^* if the following three conditions are satisfied:

(a) For each vertex v of G^*, the labels around v total to 4.

(b) Every inner facial 3-cycle C of G^* has labels $0, 1$ and 1. If C is a maximal filled 3-cycle in G, then the vertex labeled 0 in C is adjacent to all the other vertices of the inside graph $G(C)$ of C.

(c) Every outer angle has label $2, 3$ or 4.

For an inner triangulated plane graph G, Miura *et al.* obtained the following result [4].

Lemma 2.1 An inner triangulated plane graph G has an open RI-drawing if the frame graph G^* of G has a good labeling.

3 Main Theorem

In this section, we give our main theorem.

Theorem 1 An inner triangulated plane graph G has an open RI-drawing if the frame graph G^* of G has a labeling satisfying following three conditions (a)-(c):

(a) For each vertex v of G^*, the labels around v total to 4.

(b) Every inner facial 3-cycle C of G^* has labels $0, 1$ and 1. If C is a maximal filled 3-cycle in G, then the vertex labeled 0 in C is adjacent to all the other vertices of the inside graph $G(C)$ of C.

(c) Every outer angle has label $1, 2, 3$ or 4.

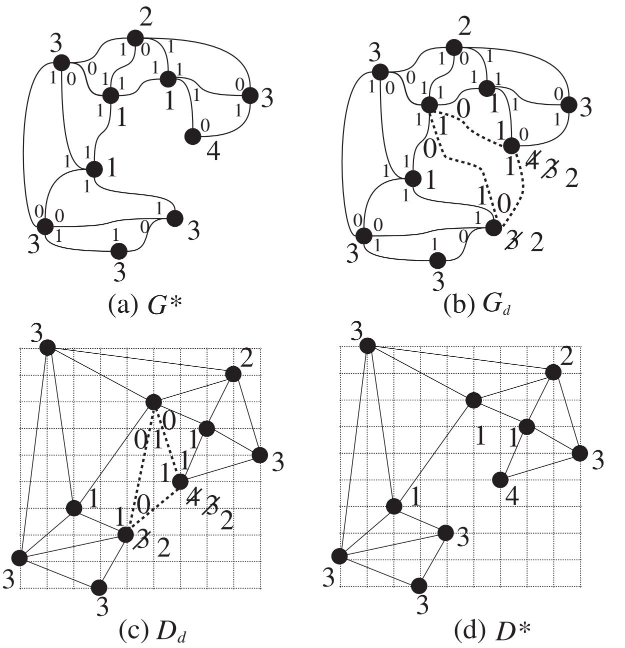
4 Outline of Our Algorithm

In this section, we outline our algorithm.

Suppose that the frame graph G^* of an inner plane graph G has a labeling satisfying Theorem 1 and a labeling is given, as illustrated in Fig. 2(a). The outline of our algorithm is as follows. We first create a new inner triangulated plane graph G_d from G^* by adding some dummy edges and renew some labels, such that G_d has a good labeling, as illustrated in Fig. 2(b). We then obtain an open rectangle-of-influence drawing D_d of G_d by using [4]'s algorithm, as illustrated in Fig. 2(c). Finally, we obtain an open rectangle-of-influence drawing D^* of G^* by deleting dummy edges from D_d, as illustrated in Fig. 2(d).

Let v_i be an outer vertex of G^* and let v_{i-1} (v_{i+1}) be the vertex preceding (succeeding) v_i on $C_o(G^*)$. Let $L(v_i)$ be the label of an outer vertex v_i. Our label renewal algorithm is as follows.

Procedure Label Renewal
begin
1 For each outer vertex v_i, find $max\{L(v_{i-1}), L(v_{i+1})\}$.
while there is a vertex v_i such that $L(v_i) = 1$ **do**
2 Let v_l be the outer vertex such that $L(v_l) = 1$ and $max\{L(v_{l-1}), L(v_{l+1})\} \geq 2$.
3 Add a dummy edge (v_{l-1}, v_{l+1}). Let F_l be an inner face created by adding a dummy edge (v_{l-1}, v_{l+1}) and let α (β) be a label of v_{l-1} (v_{l+1}) for F_l.
if $L(v_{l-1}) \geq L(v_{l+1})$ **then**
4 $L(v_{l-1}) = L(v_{l-1}) - 1$; $\alpha = 1$; $\beta = 0$;
if $L(v_{l-1}) < L(v_{l+1})$ **then**
5 $L(v_{l+1}) = L(v_{l+1}) - 1$; $\alpha = 0$; $\beta = 1$;
end.



(a) G^* (b) G_d (c) D_d (d) D^*

Figure 2: Outline of our algorithm.

We have the following Lemma.

Lemma 4.1 Algorithm *Label Renewal* finds a good labeling of G_d in linear time.

We thus have the following Theorem.

Theorem 2 One can construct an open RI-drawing of an inner triangulated plane graph G on a $W \times H \leq (n-1) \times (n-1)$ grid in linear time if G^* has a labeling satisfying Theorem 1 and a labeling is given.

References

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