A-024

Balanced (C_4, C_{18})-$2t$-Foil Decomposition Algorithm of Complete Graphs

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1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_4 and C_{18} be *the* 4-*cycle* and *the* 18-*cycle*, respectively. *The* (C_4, C_{18})-$2t$-*foil* is a graph of t edge-disjoint C_4's and t edge-disjoint C_{18}'s with a common vertex and the common vertex is called *the center of the* (C_4, C_{18})-$2t$-*foil*. In particular, the (C_4, C_{18})-2-foil is called *the* (C_4, C_{18})-*bowtie*. When K_n is decomposed into edge-disjoint sum of (C_4, C_{18})-$2t$-foils, we say that K_n *has a* (C_4, C_{18})-$2t$-*foil decomposition*. Moreover, when every vertex of K_n appears in the same number of (C_4, C_{18})-$2t$-foils, we say that K_n *has a balanced* (C_4, C_{18})-$2t$-*foil decomposition* and this number is called *the replication number*.
Note that (C_4, C_{18})-$2t$-foil has $20t + 1$ vertices and $22t$ edges.

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or 3 (mod 6). This decomposition is known as *a Steiner triple system*. See Colbourn and Rosa[2] and Wallis[15]. Horák and Rosa[3] proved that K_n has a (C_3, C_3)-bowtie decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a bowtie system*.
In this sense, our balanced (C_4, C_{18})-$2t$-foil decomposition of K_n is to be known as *a balanced* (C_4, C_{18})-$2t$-*foil system*.

2. Balanced (C_4, C_{18})-$2t$-foil decomposition of K_n

Theorem. K_n has a balanced (C_4, C_{18})-$2t$-foil decomposition if and only if $n \equiv 1$ (mod $44t$).
Proof. (Necessity) Suppose that K_n has a balanced (C_4, C_{18})-$2t$-foil decomposition. Let b be the number of (C_4, C_{18})-$2t$-foils and r be the replication number. Then $b = n(n-1)/44t$ and $r = (20t+1)(n-1)/44t$. Among r (C_4, C_{18})-$2t$-foils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_4, C_{18})-$2t$-foils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4tr_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/44t$ and $r_2 = 20(n-1)/44$. Therefore, $n \equiv 1$ (mod $44t$) is necessary.

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(Sufficiency) Put $n = 44st + 1$ and $T = st$. Then $n = 44T + 1$.
Construct a (C_4, C_{18})-$2T$-foil as follows:
$\{(44T+1, 36T+1, 27T+1, 37T+1), (44T+1, 1, 2T+2, 13T+2, 18T+3, 32T+3, 6T+3, 24T+3, 38T+4, 17T+3, 40T+4, 25T+3, 8T+3, 33T+3, 20T+3, 16T+2, 4T+2, T+1)\} \cup$
$\{(44T+1, 36T+2, 27T+3, 37T+2), (44T+1, 2, 2T+4, 13T+3, 18T+5, 32T+4, 6T+5, 24T+4, 38T+6, 17T+4, 40T+6, 25T+4, 8T+5, 33T+4, 20T+5, 16T+3, 4T+4, T+2)\} \cup$
$\{(44T+1, 36T+3, 27T+5, 37T+3), (44T+1, 3, 2T+6, 13T+4, 18T+7, 32T+5, 6T+7, 24T+5, 38T+8, 17T+5, 40T+8, 25T+5, 8T+7, 33T+5, 20T+7, 16T+4, 4T+6, T+3)\} \cup ... \cup$
$\{(44T+1, 37T, 29T-1, 38T), (44T+1, T, 4T, 14T+1, 20T+1, 33T+2, 8T+1, 25T+2, 40T+2, 18T+2, 42T+2, 26T+2, 10T+1, 34T+2, 22T+1, 17T+1, 6T, 2T)\}$. ($22T$ edges, $22T$ all lengths)
Decompose the (C_4, C_{18})-$2T$-foil into s (C_4, C_{18})-$2t$-foils. Then these s starters comprise a balanced (C_4, C_{18})-$2t$-foil decomposition of K_n.

Corollary. K_n has a balanced (C_4, C_{18})-bowtie decomposition if and only if $n \equiv 1$ (mod 44).

Example 1. A (C_4, C_{18})-2-foil of K_{45}.
$\{(45, 37, 28, 38), (45, 1, 4, 15, 21, 35, 9, 27, 42, 20, 44, 28, 11, 36, 23, 18, 6, 2)\}$.
(22 edges, 22 all lengths)
This starter comprises a balanced (C_4, C_{18})-2-foil decomposition of K_{45}.

Example 2. A (C_4, C_{18})-4-foil of K_{89}.
$\{(89, 73, 55, 75), (89, 1, 6, 28, 39, 67, 15, 51, 80, 37, 84, 53, 19, 69, 43, 34, 10, 3)\} \cup$
$\{(89, 74, 57, 76), (89, 2, 8, 29, 41, 68, 17, 52, 82, 38, 86, 54, 21, 70, 45, 35, 12, 4)\}$.
(44 edges, 44 all lengths)
This starter comprises a balanced (C_4, C_{18})-4-foil decomposition of K_{89}.

Example 3. A (C_4, C_{18})-6-foil of K_{133}.
$\{(133, 109, 82, 112), (133, 1, 8, 41, 57, 99, 21, 75, 118, 54, 124, 78, 27, 102, 63, 50, 14, 4)\} \cup$
$\{(133, 110, 84, 113), (133, 2, 10, 42, 59, 100, 23, 76, 120, 55, 126, 79, 29, 103, 65, 51, 16, 5)\} \cup$
$\{(133, 111, 86, 114), (133, 3, 12, 43, 61, 101, 25, 77, 122, 56, 128, 80, 31, 104, 67, 52, 18, 6)\}$.
(66 edges, 66 all lengths)

This starter comprises a balanced (C_4, C_{18})-6-foil decomposition of K_{133}.

Example 4. A (C_4, C_{18})-8-foil of K_{177}.
$\{(177, 145, 109, 149), (177, 1, 10, 54, 75, 131, 27, 99, 156, 71, 164, 103, 35, 135, 83, 66, 18, 5)\} \cup$
$\{(177, 146, 111, 150), (177, 2, 12, 55, 77, 132, 29, 100, 158, 72, 166, 104, 37, 136, 85, 67, 20, 6)\} \cup$
$\{(177, 147, 113, 151), (177, 3, 14, 56, 79, 133, 31, 101, 160, 73, 168, 105, 39, 137, 87, 68, 22, 7)\} \cup$
$\{(177, 148, 115, 152), (177, 4, 16, 57, 81, 134, 33, 102, 162, 74, 170, 106, 41, 138, 89, 69, 24, 8)\}$.
(88 edges, 88 all lengths)
This starter comprises a balanced (C_4, C_{18})-8-foil decomposition of K_{177}.

Example 5. A (C_4, C_{18})-10-foil of K_{221}.
$\{(221, 181, 136, 186), (221, 1, 12, 67, 93, 163, 33, 123, 194, 88, 204, 128, 43, 168, 103, 82, 22, 6)\} \cup$
$\{(221, 182, 138, 187), (221, 2, 14, 68, 95, 164, 35, 124, 196, 89, 206, 129, 45, 169, 105, 83, 24, 7)\} \cup$
$\{(221, 183, 140, 188), (221, 3, 16, 69, 97, 165, 37, 125, 198, 90, 208, 130, 47, 170, 107, 84, 26, 8)\} \cup$
$\{(221, 184, 142, 189), (221, 4, 18, 70, 99, 166, 39, 126, 200, 91, 210, 131, 49, 171, 109, 85, 28, 9)\} \cup$
$\{(221, 185, 144, 190), (221, 5, 20, 71, 101, 167, 41, 127, 202, 92, 212, 132, 51, 172, 111, 86, 30, 10)\}$.
(110 edges, 110 all lengths)
This starter comprises a balanced (C_4, C_{18})-10-foil decomposition of K_{221}.

Example 6. A (C_4, C_{18})-12-foil of K_{265}.
$\{(265, 217, 163, 223), (265, 1, 14, 80, 111, 195, 39, 147, 232, 105, 244, 153, 51, 201, 123, 98, 26, 7)\} \cup$
$\{(265, 218, 165, 224), (265, 2, 16, 81, 113, 196, 41, 148, 234, 106, 246, 154, 53, 202, 125, 99, 28, 8)\} \cup$
$\{(265, 219, 167, 225), (265, 3, 18, 82, 115, 197, 43, 149, 236, 107, 248, 155, 55, 203, 127, 100, 30, 9)\} \cup$
$\{(265, 220, 169, 226), (265, 4, 20, 83, 117, 198, 45, 150, 238, 108, 250, 156, 57, 204, 129, 101, 32, 10)\} \cup$
$\{(265, 221, 171, 227), (265, 5, 22, 84, 119, 199, 47, 151, 240, 109, 252, 157, 59, 205, 131, 102, 34, 11)\} \cup$
$\{(265, 222, 173, 228), (265, 6, 24, 85, 121, 200, 49, 152, 242, 110, 254, 158, 61, 206, 133, 103, 36, 12)\}$.
(132 edges, 132 all lengths)
This starter comprises a balanced (C_4, C_{18})-12-foil decomposition of K_{265}.

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