Balanced (C_5, C_{20})-$2t$-Foil Decomposition Algorithm of Complete Graphs

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1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_5 and C_{20} be *the* 5*-cycle* and *the* 20*-cycle*, respectively. *The* (C_5, C_{20})*-*$2t$*-foil* is a graph of t edge-disjoint C_5's and t edge-disjoint C_{20}'s with a common vertex and the common vertex is called *the center of the* (C_5, C_{20})-$2t$*-foil*. In particular, the (C_5, C_{20})-2-foil is called *the* (C_5, C_{20})*-bowtie*. When K_n is decomposed into edge-disjoint sum of (C_5, C_{20})-$2t$-foils, we say that K_n *has a* (C_5, C_{20})*-*$2t$*-foil decomposition*. Moreover, when every vertex of K_n appears in the same number of (C_5, C_{20})-$2t$-foils, we say that K_n *has a balanced* (C_5, C_{20})*-*$2t$*-foil decomposition* and this number is called *the replication number*.
Note that (C_5, C_{20})-$2t$-foil has $23t + 1$ vertices and $25t$ edges.

2. Balanced (C_4, C_{18})-$2t$-foil decomposition of K_n

Theorem. K_n has a balanced (C_5, C_{20})-$2t$-foil decomposition if and only if $n \equiv 1 \pmod{50t}$.

Proof. **(Necessity)** Suppose that K_n has a balanced (C_5, C_{20})-$2t$-foil decomposition. Let b be the number of (C_5, C_{20})-$2t$-foils and r be the replication number. Then $b = n(n-1)/50t$ and $r = (23t+1)(n-1)/50t$. Among r (C_5, C_{20})-$2t$-foils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_5, C_{20})-$2t$-foils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4tr_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/50t$ and $r_2 = 23(n-1)/50$. Therefore, $n \equiv 1 \pmod{50t}$ is necessary.
(Sufficiency) Put $n = 50st + 1$ and $T = st$. Then $n = 50T + 1$.
Case 1. n = 51. (Example 1. Balanced (C_5, C_{20})-2-foil decomposition of K_{51}.)
Case 2. $\mathbf{n = 50T + 1, T \geq 2}$. Construct a (C_5, C_{20})-$2T$-foil as follows:
$\{(50T+1, 1, 22T+2, 46T+2, 22T), (50T+1, 3T+1, 4T+2, 12T+2, 22T+3, 29T+3, 44T+4, 6T+3, 26T+4, 41T+4, 9T+4, T+3, 33T+4, 17T+3, 7T+3, 38T+3, 24T+3, 19T+2, 14T+2, 2T+1)\}$
$\cup$
$\{(50T+1, 2, 22T+4, 46T+3, 22T-1), (50T+1, 3T+2, 4T+4, 12T+3, 22T+5, 29T+4, 44T+6, 6T+4, 26T+6, 41T+5, 9T+6, T+4, 33T+6, 17T+4, 7T+5, 38T+4, 24T+5, 19T+3, 14T+4, 2T+2)\}$
$\cup$
$\{(50T+1, 3, 22T+6, 46T+4, 22T-2), (50T+1, 3T+3, 4T+6, 12T+4, 22T+7, 29T+5, 44T+8, 6T+5, 26T+8, 41T+6, 9T+8, T+5, 33T+8, 17T+5, 7T+7, 38T+5, 24T+7, 19T+4, 14T+6, 2T+3)\}$
$\cup$
...
$\cup$
$\{(50T+1, T-2, 24T-4, 47T-1, 21T+3), (50T+1, 4T-2, 6T-4, 13T-1, 24T-3, 30T, 46T-2, 7T, 28T-2, 42T+1, 11T-2, 2T, 35T-2, 18T, 9T-3, 39T, 26T-3, 20T-1, 16T-4, 3T-2)\} \cup$
$\{(50T+1, T-1, 24T-2, 47T, 21T+2), (50T+1, 4T-1, 6T-2, 13T, 24T-1, 30T+1, 46T, 7T+1, 28T, 42T+2, 11T, 44T, 35T, 18T+1, 9T-1, 39T+1, 26T-1, 20T, 16T-2, 3T-1)\} \cup$
$\{(50T+1, T, 24T, 47T+1, 21T+1), (50T+1, 4T, 6T, 13T+1, 24T+1, 30T+2, 41T+3, 7T+2, 28T+2, 47T+2, 11T+2, 44T+1, 35T+2, 18T+2, 9T+1, 39T+2, 26T+1, 20T+1, 16T, 3T)\}$.
($25T$ edges, $25T$ all lengths)
Decompose the (C_5, C_{20})-$2T$-foil into s (C_5, C_{20})-$2t$-foils. Then these s starters comprise a balanced (C_5, C_{20})-$2t$-foil decomposition of K_n.

Corollary. K_n has a balanced (C_5, C_{20})-bowtie decomposition if and only if $n \equiv 1 \pmod{50}$.

Example 1. Balanced (C_5, C_{20})-2-foil decomposition of K_{51}.
$\{(51, 1, 24, 48, 22), (51, 4, 6, 14, 25, 32, 44, 9, 30, 49, 13, 46, 37, 20, 10, 41, 27, 21, 16, 3)\}$.
(25 edges, 25 all lengths)
This starter comprises a balanced (C_5, C_{20})-2-foil decomposition of K_{51}.

Example 2. Balanced (C_5, C_{20})-4-foil decomposition of K_{101}.
$\{(101, 1, 46, 94, 44), (101, 7, 10, 26, 47, 61, 92, 15, 56, 86, 22, 88, 70, 37, 17, 79, 51, 40, 30, 5)\} \cup$
$\{(101, 2, 48, 95, 43), (101, 8, 12, 27, 49, 62, 85, 16, 58, 96, 24, 89, 72, 38, 19, 80, 53, 41, 32, 6)\}$.
(50 edges, 50 all lengths)
This starter comprises a balanced (C_5, C_{20})-4-foil decomposition of K_{101}.

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Example 3. Balanced (C_5, C_{20})-6-foil decomposition of K_{151}.
$\{(151,1,68,140,66),(151,10,14,38,69,90,136,21,82,127,31,6,103,54,24,117,75,59,44,7)\} \cup$
$\{(151,2,70,141,65),(151,11,16,39,71,91,138,22,84,128,33,132,105,55,26,118,77,60,46,8)\} \cup$
$\{(151,3,72,142,64),(151,12,18,40,73,92,126,23,86,143,35,133,107,56,28,119,79,61,48,9)\}$.
(75 edges, 75 all lengths)
This starter comprises a balanced (C_5, C_{20})-6-foil decomposition of K_{151}.

Example 4. Balanced (C_5, C_{20})-8-foil decomposition of K_{201}.
$\{(201,1,90,186,88),(201,13,18,50,91,119,180,27,108,168,40,7,136,71,31,155,99,78,58,9)\} \cup$
$\{(201,2,92,187,87),(201,14,20,51,93,120,182,28,110,169,42,8,138,72,33,156,101,79,60,10)\} \cup$
$\{(201,3,94,188,86),(201,15,22,52,95,121,184,29,112,170,44,176,140,73,35,157,103,80,62,11)\} \cup$
$\{(201,4,96,189,85),(201,16,24,53,97,122,167,30,114,190,46,177,142,74,37,158,105,81,64,12)\}$.
(100 edges, 100 all lengths)
This starter comprises a balanced (C_5, C_{20})-8-foil decomposition of K_{201}.

Example 5. Balanced (C_5, C_{20})-10-foil decomposition of K_{251}.
$\{(251,1,112,232,110),(251,16,22,62,113,148,224,33,134,209,49,8,169,88,38,193,123,97,72,11)\} \cup$
$\{(251,2,114,233,109),(251,17,24,63,115,149,226,34,136,210,51,9,171,89,40,194,125,98,74,12)\} \cup$
$\{(251,3,116,234,108),(251,18,26,64,117,150,228,35,138,211,53,10,173,90,42,195,127,99,76,13)\} \cup$
$\{(251,4,118,235,107),(251,19,28,65,119,151,230,36,140,212,55,220,175,91,44,196,129,100,78,14)\} \cup$
$\{(251,5,120,236,106),(251,20,30,66,121,152,208,37,142,237,57,221,177,92,46,197,131,101,80,15)\}$.
(125 edges, 125 all lengths)
This starter comprises a balanced (C_5, C_{20})-10-foil decomposition of K_{251}.

Example 6. Balanced (C_5, C_{20})-12-foil decomposition of K_{301}.
$\{(301,1,134,278,132),(301,19,26,74,135,177,268,39,160,250,58,9,202,105,45,231,147,116,86,13)\} \cup$
$\{(301,2,136,279,131),(301,20,28,75,137,178,270,40,162,251,60,10,204,106,47,232,149,117,88,14)\} \cup$
$\{(301,3,138,280,130),(301,21,30,76,139,179,272,41,164,252,62,11,206,107,49,233,151,118,90,15)\} \cup$
$\{(301,4,140,281,129),(301,22,32,77,141,180,274,42,166,253,64,12,208,108,51,234,153,119,92,16)\} \cup$
$\{(301,5,142,282,128),(301,23,34,78,143,181,276,43,168,254,66,264,210,109,53,235,155,120,94,17)\} \cup$
$\{(301,6,144,283,127),(301,24,36,79,145,182,249,44,170,284,68,265,212,110,55,236,157,121,96,18)\}$.
(150 edges, 150 all lengths)
This starter comprises a balanced (C_5, C_{20})-12-foil decomposition of K_{301}.

Example 7. Balanced (C_5, C_{20})-14-foil decomposition of K_{351}.
$\{(351,1,156,324,154),(351,22,30,86,157,206,312,45,186,291,67,10,235,122,52,269,171,135,100,15)\} \cup$
$\{(351,2,158,325,153),(351,23,32,87,159,207,314,46,188,292,69,11,237,123,54,270,173,136,102,16)\} \cup$
$\{(351,3,160,326,152),(351,24,34,88,161,208,316,47,190,293,71,12,239,124,56,271,175,137,104,17)\} \cup$
$\{(351,4,162,327,151),(351,25,36,89,163,209,318,48,192,294,73,13,241,125,58,272,177,138,106,18)\} \cup$
$\{(351,5,164,328,150),(351,26,38,90,165,210,320,49,194,295,75,14,243,126,60,273,179,139,108,19)\} \cup$
$\{(351,6,166,329,149),(351,27,40,91,167,211,322,50,196,296,77,308,245,127,62,274,181,140,110,20)\} \cup$
$\{(351,7,168,330,148),(351,28,42,92,169,212,290,51,198,331,79,309,247,128,64,275,183,141,112,21)\}$.
(175 edges, 175 all lengths)
This starter comprises a balanced (C_5, C_{20})-14-foil decomposition of K_{351}.

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