A-002

Balanced (C_4, C_{12})-$2t$-Foil Decomposition Algorithm of Complete Graphs

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1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_4 and C_{12} be *the* 4-*cycle* and *the* 12-*cycle*, respectively. *The* (C_4, C_{12})-$2t$-*foil* is a graph of t edge-disjoint C_4's and t edge-disjoint C_{12}'s with a common vertex and the common vertex is called *the center of the* (C_4, C_{12})-$2t$-*foil*. In particular, the (C_4, C_{12})-2-foil is called *the* (C_4, C_{12})-*bowtie*. When K_n is decomposed into edge-disjoint sum of (C_4, C_{12})-$2t$-foils, we say that K_n *has a* (C_4, C_{12})-$2t$-*foil decomposition*. Moreover, when every vertex of K_n appears in the same number of (C_4, C_{12})-$2t$-foils, we say that K_n *has a balanced* (C_4, C_{12})-$2t$-*foil decomposition* and this number is called *the replication number*.
Note that (C_4, C_{12})-$2t$-foil has $14t + 1$ vertices and $16t$ edges.

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or 3 (mod 6). This decomposition is known as *a Steiner triple system*. See Colbourn and Rosa[2] and Wallis[15]. Horák and Rosa[3] proved that K_n has a (C_3, C_3)-bowtie decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a bowtie system*.
In this sense, our balanced (C_4, C_{12})-$2t$-foil decomposition of K_n is to be known as *a balanced* (C_4, C_{12})-$2t$-*foil system*.

2. Balanced (C_4, C_{12})-$2t$-foil decomposition of K_n

Theorem. K_n has a balanced (C_4, C_{12})-$2t$-foil decomposition if and only if $n \equiv 1 \pmod{32t}$.

Proof. (Necessity) Suppose that K_n has a balanced (C_4, C_{12})-$2t$-foil decomposition. Let b be the number of (C_4, C_{12})-$2t$-foils and r be the replication number. Then $b = n(n-1)/32t$ and $r = (14t+1)(n-1)/32t$. Among r (C_4, C_{12})-$2t$-foils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_4, C_{12})-$2t$-foils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4tr_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/32t$ and $r_2 = 14(n-1)/32$. Therefore, $n \equiv 1 \pmod{32t}$ is necessary.
(Sufficiency) Put $n = 32st + 1$, $T = st$. Then $n = 32T + 1$.
Construct a (C_4, C_{12})-$2T$-foil as follows:
$\{(32T+1, 2T+1, 9T+2, 3T+1), (32T+1, 1, 4T+2, 14T+2, 24T+3, 12T+2, 29T+3, 13T+2, 26T+3, 15T+2, 6T+2, T+1)\} \cup$
$\{(32T+1, 2T+2, 9T+4, 3T+2), (32T+1, 2, 4T+4, 14T+3, 24T+5, 12T+3, 29T+5, 13T+3, 26T+5, 15T+3, 6T+4, T+2)\} \cup$
$\{(32T+1, 2T+3, 9T+6, 3T+3), (32T+1, 3, 4T+6, 14T+4, 24T+7, 12T+4, 29T+7, 13T+4, 26T+7, 15T+4, 6T+6, T+3)\} \cup ... \cup$
$\{(32T+1, 3T, 11T, 4T), (32T+1, T, 6T, 15T+1, 26T+1, 13T+1, 31T+1, 14T+1, 28T+1, 16T+1, 8T, 2T)\}$.
($16T$ edges, $16T$ all lengths)
Decompose the (C_4, C_{12})-$2T$-foil into s (C_4, C_{12})-$2t$-foils. Then these s starters comprise a balanced (C_4, C_{12})-$2t$-foil decomposition of K_n.

Corollary. K_n has a balanced (C_4, C_{12})-bowtie decomposition if and only if $n \equiv 1 \pmod{32}$.

Example 1. A (C_4, C_{12})-2-foil of K_{33}.
$\{(33, 3, 11, 4), (33, 1, 6, 16, 27, 14, 32, 15, 29, 17, 8, 2)\}$.
(16 edges, 16 all lengths)
This starter comprises a balanced (C_4, C_{12})-2-foil decomposition of K_{33}.

Example 2. A (C_4, C_{12})-4-foil of K_{65}.
$\{(65, 5, 20, 7), (65, 1, 10, 30, 51, 26, 61, 28, 55, 32, 14, 3)\} \cup$
$\{(65, 6, 22, 8), (65, 2, 12, 31, 53, 27, 63, 29, 57, 33, 16, 4)\}$.
(32 edges, 32 all lengths)
This starter comprises a balanced (C_4, C_{12})-4-foil decomposition of K_{65}.

Example 3. A (C_4, C_{12})-6-foil of K_{97}.
$\{(97, 7, 29, 10), (97, 1, 14, 44, 75, 38, 90, 41, 81, 47, 20, 4)\}$
$\cup$
$\{(97, 8, 31, 11), (97, 2, 16, 45, 77, 39, 92, 42, 83, 48, 22, 5)\}$
$\cup$
$\{(97, 9, 33, 12), (97, 3, 18, 46, 79, 40, 94, 43, 85, 49, 24, 6)\}$.
(48 edges, 48 all lengths)
This starter comprises a balanced (C_4, C_{12})-6-foil decomposition of K_{97}.

Example 4. A (C_4, C_{12})-8-foil of K_{129}.
$\{(129, 9, 38, 13), (129, 1, 18, 58, 99, 50, 119, 54, 107, 62, 26, 5)\}$
$\cup$
$\{(129, 10, 40, 14), (129, 2, 20, 59, 101, 51, 121, 55, 109, 63, 28, 6)\}$
$\cup$
$\{(129, 11, 42, 15), (129, 3, 22, 60, 103, 52, 123, 56, 111, 64, 30, 7)\}$
$\cup$
$\{(129, 12, 44, 16), (129, 4, 24, 61, 105, 53, 125, 57, 113, 65, 32, 8)\}$.
(64 edges, 64 all lengths)

This starter comprises a balanced (C_4, C_{12})-8-foil decomposition of K_{129}.

Example 5. A (C_4, C_{12})-10-foil of K_{161}.
$\{(161, 11, 47, 16), (161, 1, 22, 72, 123, 62, 148, 67, 133, 77, 32, 6)\}$
$\cup$
$\{(161, 12, 49, 17), (161, 2, 24, 73, 125, 63, 150, 68, 135, 78, 34, 7)\}$
$\cup$
$\{(161, 13, 51, 18), (161, 3, 26, 74, 127, 64, 152, 69, 137, 79, 36, 8)\}$
$\cup$
$\{(161, 14, 53, 19), (161, 4, 28, 75, 129, 65, 154, 70, 139, 80, 38, 9)\}$
$\cup$
$\{(161, 15, 55, 20), (161, 5, 30, 76, 131, 66, 156, 71, 141, 81, 40, 10)\}$.
(80 edges, 80 all lengths)

This starter comprises a balanced (C_4, C_{12})-10-foil decomposition of K_{161}.

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