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Balanced C_{16}-Bowtie Decomposition Algorithm of Complete Graphs

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1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_{16} be *the cycle* on 16 vertices. *The C_{16}-bowtie* is a graph of 2 edge-disjoint C_{16}'s with a common vertex and the common vertex is called *the center of the C_{16}-bowtie*. When K_n is decomposed into edge-disjoint sum of C_{16}-bowties, it is called that *K_n has a C_{16}-bowtie decomposition.* Moreover, when every vertex of K_n appears in the same number of C_{16}-bowties, it is called that *K_n has a balanced C_{16}-bowtie decomposition* and this number is called *the replication number.*

In this paper, it is shown that the necessary and sufficient condition for the existence of a balanced C_{16}-bowtie decomposition of K_n is $n \equiv 1 \pmod{64}$.

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or $3 \pmod 6$. This decomposition is known as *a Steiner triple system*. See Colbourn and Rosa[1] and Wallis[6, Chapter 12 : Triple Systems]. Horák and Rosa[2] proved that K_n has a C_3-bowtie decomposition if and only if $n \equiv 1$ or $9 \pmod{12}$. This decomposition is known as *a C_3-bowtie system.* In this sense, our balanced C_{16}-bowtie decomposition of K_n is to be known as *a balanced C_{16}-bowtie system.*

2. Balanced C_{16}-bowtie decomposition of K_n

Notation. We denote a C_{16}-bowtie passing through $v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10} - v_{11} - v_{12} - v_{13} - v_{14} - v_{15} - v_{16} - v_1$, $v_1 - v_{17} - v_{18} - v_{19} - v_{20} - v_{21} - v_{22} - v_{23} - v_{24} - v_{25} - v_{26} - v_{27} - v_{28} - v_{29} - v_{30} - v_{31} - v_1$ by $\{(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}), (v_1, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}, v_{28}, v_{29}, v_{30}, v_{31})\}$.

Theorem. K_n has a balanced C_{16}-bowtie decomposition if and only if $n \equiv 1 \pmod{64}$.

Proof. (Necessity) Suppose that K_n has a balanced C_{16}-bowtie decomposition. Let b be the number of C_{16}-bowties and r be the replication number. Then $b = n(n-1)/64$ and $r = 31(n-1)/64$. Among r C_{16}-bowties having a vertex v of K_n, let r_1 and r_2 be the numbers of C_{16}-bowties in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4r_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/64$ and $r_2 = 30(n-1)/64$. Therefore, $n \equiv 1 \pmod{64}$ is necessary.

(Sufficiency) $n = 64t + 1$. Construct tn C_{16}-bowties as follows:

$B_i^{(1)} = \{ (i, i+1, i+4t+2, i+24t+2, i+36t+3, i+48t+3, i+4t+3, i+44t+3, i+14t+3, i+46t+3, i+8t+3, i+50t+3, i+40t+3, i+26t+2, i+8t+2, i+2t+1), (i, i+2, i+4t+4, i+24t+3, i+36t+5, i+48t+4, i+4t+5, i+44t+4, i+14t+5, i+46t+4, i+8t+5, i+50t+4, i+40t+5, i+26t+3, i+8t+4, i+2t+2) \}$

$B_i^{(2)} = \{ (i, i+3, i+4t+6, i+24t+4, i+36t+7, i+48t+5, i+4t+7, i+44t+5, i+14t+7, i+46t+5, i+8t+7, i+50t+5, i+40t+7, i+26t+4, i+8t+6, i+2t+3), (i, i+4, i+4t+8, i+24t+5, i+36t+9, i+48t+6, i+4t+9, i+44t+6, i+14t+9, i+46t+6, i+8t+9, i+50t+6, i+40t+9, i+26t+5, i+8t+8, i+2t+4) \}$

...

$B_i^{(t)} = \{ (i, i+2t-1, i+8t-2, i+26t, i+40t-1, i+50t+1, i+8t-1, i+46t+1, i+18t-1, i+48t+1, i+12t-1, i+52t+1, i+44t-1, i+28t, i+12t-2, i+4t-1), (i, i+2t, i+8t, i+26t+1, i+40t+1, i+50t+2, i+8t+1, i+46t+2, i+18t+1, i+48t+2, i+12t+1, i+52t+2, i+44t+1, i+28t+1, i+12t, i+4t) \}$

$(i = 1, 2, ..., n)$,

where the additions $i + x$ are taken modulo n

with residues $1, 2, ..., n$.
Then they comprise a balanced C_{16}-bowtie decomposition of K_n.

Example 1. Balanced C_{16}-bowtie decomposition of K_{65}.

$B_i = \{(i, i+1, i+6, i+26, i+39, i+51, i+7, i+47, i+17, i+49, i+11, i+53, i+43, i+28, i+10, i+3),$
$(i, i+2, i+8, i+27, i+41, i+52, i+9, i+48, i+19, i+50, i+13, i+54, i+45, i+29, i+12, i+4)\}$
$(i = 1, 2, ..., 65).$

Example 2. Balanced C_{16}-bowtie decomposition of K_{129}.

$B_i^{(1)} = \{(i, i+1, i+10, i+50, i+75, i+99, i+11, i+91, i+31, i+95, i+19, i+103, i+83, i+54, i+18, i+5),$
$(i, i+2, i+12, i+51, i+77, i+100, i+13, i+92, i+33, i+96, i+21, i+104, i+85, i+55, i+20, i+6)\}$
$B_i^{(2)} = \{(i, i+3, i+14, i+52, i+79, i+101, i+15, i+93, i+35, i+97, i+23, i+105, i+87, i+56, i+22, i+7),$
$(i, i+4, i+15, i+53, i+81, i+102, i+17, i+94, i+37, i+98, i+25, i+106, i+89, i+57, i+24, i+8)\}$
$(i = 1, 2, ..., 129).$

Example 3. Balanced C_{16}-bowtie decomposition of K_{193}.

$B_i^{(1)} = \{(i, i+1, i+14, i+74, i+111, i+147, i+15, i+135, i+45, i+141, i+27, i+153, i+123, i+80, i+26, i+7),$
$(i, i+2, i+16, i+75, i+113, i+148, i+17, i+136, i+47, i+142, i+29, i+154, i+125, i+81, i+28, i+8)\}$
$B_i^{(2)} = \{(i, i+3, i+18, i+76, i+115, i+149, i+19, i+137, i+49, i+143, i+31, i+155, i+127, i+82, i+30, i+9),$
$(i, i+4, i+20, i+77, i+117, i+150, i+21, i+138, i+51, i+144, i+33, i+156, i+129, i+83, i+32, i+10)\}$
$B_i^{(3)} = \{(i, i+5, i+22, i+78, i+119, i+151, i+23, i+139, i+53, i+145, i+35, i+157, i+131, i+84, i+34, i+11),$
$(i, i+6, i+24, i+79, i+121, i+152, i+25, i+140, i+55, i+146, i+37, i+158, i+133, i+85, i+36, i+12)\}$
$(i = 1, 2, ..., 193).$

Example 4. Balanced C_{16}-bowtie decomposition of K_{257}.

$B_i^{(1)} = \{(i, i+1, i+18, i+98, i+147, i+195, i+19, i+179, i+59, i+187, i+35, i+203, i+163, i+106, i+34, i+9),$
$(i, i+2, i+20, i+99, i+149, i+196, i+21, i+180, i+61, i+188, i+37, i+204, i+165, i+107, i+36, i+10)\}$
$B_i^{(2)} = \{(i, i+3, i+22, i+100, i+151, i+197, i+23, i+181, i+63, i+189, i+39, i+205, i+167, i+108, i+38, i+11),$
$(i, i+4, i+24, i+101, i+153, i+198, i+25, i+182, i+65, i+190, i+41, i+206, i+169, i+109, i+40, i+12)\}$
$B_i^{(3)} = \{(i, i+5, i+26, i+102, i+155, i+199, i+27, i+183, i+67, i+191, i+43, i+207, i+171, i+110, i+42, i+13),$
$(i, i+6, i+28, i+103, i+157, i+200, i+29, i+184, i+69, i+192, i+45, i+208, i+173, i+111, i+44, i+14)\}$
$B_i^{(4)} = \{(i, i+7, i+30, i+104, i+159, i+201, i+31, i+185, i+71, i+193, i+47, i+209, i+175, i+112, i+46, i+15),$
$(i, i+8, i+32, i+105, i+161, i+202, i+33, i+186, i+73, i+194, i+49, i+210, i+177, i+113, i+48, i+16)\}$
$(i = 1, 2, ..., 257).$

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