CA-003

Selections and Fixed Points of $3M$ Upper Caratheodory Correspondences

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1 Introduction

We establish a new fixed point result for measurable-selection-valued correspondences with nonconvex and possibly disconnected values induced by upper Caratheodory correspondences (i.e., a uc correspondences). We show that if the upper semicontinuous part of a uc correspondence is *approximable* (via continuous functions), then the induced measurable-selection-valued correspondence has fixed points in its set of measurable selections. Also, we show that if the uc correspondence has, in each state, an upper semicontinuous part containing a R_δ-valued (equivalently, contractible-valued or dendritically-valued) upper semicontinuous sub-correspondence, then the upper semicontinuous part is, state-by-state, approximable - implying that the uc correspondence is Caratheodory approximable (via Caratheodory functions). Are there uc correspondences having such properties? We show that if the underlying uc correspondence has the $3M$ property - a property introduced in Page and Resende (2011) - then it will contain a minimal uc correspondence having the property that in each state the upper semicontinuous part of this minimal uc correspondence is an R_δ-valued minimal USCO.

An excellent example of a $3M$ upper Caratheodory correspondence is the Nash payoff correspondence belonging to any m-person, noncooperative parameterized state-contingent game (i.e., belonging to any $\mathcal{PSG}$). Under mild conditions on primitives, Page and Fu (2018) have shown that the Nash payoff correspondence belonging to any $\mathcal{PSG}$ is $3M$, and as a consequence, any $\mathcal{PSG}$ has an *approximable Nash payoff correspondence* - and therefore, has a *Nash payoff selection correspondence having fixed points.*

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2 Primitives of the Fixed Point Problem

To begin, suppose (Ω, B_Ω, μ) is a probability space of states, ω, with state space, Ω, a complete separable metric space, equipped with Borel σ-field, B_Ω, and probability measure, μ. Let $D := \{1, 2, 3, \ldots, m\}$ be a finite index set and for each $d \in D$, let $Y_d := [-M, M] \subset R$ be the closed bounded interval. For each $d \in D$, the interval Y_d contains the range of the functions in the function space, $\mathcal{L}^\infty_{Y_d}$. Here, $\mathcal{L}^\infty_{Y_d}$ consists of all μ-equivalence classes of real-valued measurable functions, v_d, defined on Ω with values in Y_d a.e. $[\mu]$. Because $\mathcal{L}^\infty_{Y_d}$ is weak star compact and metrizable as well as convex and locally convex (and therefore connected and locally connected - and therefore a Peano continuum), $\mathcal{L}^\infty_{Y_d}$ can be equipped with an M-convex metric $\rho_{\mathcal{L}^\infty_{Y_d}}$ compatible with the weak star topology inherited from $\mathcal{L}^\infty_R$ (the Banach space of all essentially bounded, real-valued measurable functions). The weak star topology in $\mathcal{L}^\infty_R$ is denoted by w^*_d or by $\sigma(\mathcal{L}^\infty_R, \mathcal{L}^1_R)$. Now let

$$\mathcal{L}^\infty_Y := \mathcal{L}^\infty_{Y_1} \times \cdots \times \mathcal{L}^\infty_{Y_m},$$

where $Y := Y_1 \times \cdots \times Y_m$. Equip $\mathcal{L}^\infty_Y$ with the M-convex sum metric

$$\rho_{\mathcal{L}^\infty_Y} := \sum_{d=1}^{m} \rho_{\mathcal{L}^\infty_{Y_d}}.$$

Note that $(\mathcal{L}^\infty_Y, \rho_{\mathcal{L}^\infty_Y})$ is an M-convex compact metric space of Y-valued measurable functions, and therefore, $\mathcal{L}^\infty_Y$ has a Borel σ-field, $B_{\mathcal{L}^\infty_Y}$, generated by its $\rho_{\mathcal{L}^\infty_Y}$-open sets.

Consider the upper Caratheodory correspondence,

$$\mathcal{P}(\cdot,\cdot) : \Omega \times \mathcal{L}^\infty_Y \longrightarrow 2^Y,$$

jointly measurable in $(\omega, v) \in \Omega \times \mathcal{L}^\infty_Y$ and upper semicontinuous in v for each ω, taking nonempty, ρ_Y-closed (and hence, ρ_Y-compact) values in Y, a closed, bounded convex subset of R^m. Equip Y with the sum metric,

$$\rho_Y := \sum_{d=1}^{m} \rho_{Y_d}$$

where for each d, ρ_{Y_d} is the absolute value metric on $[-M, M]$. Finally, let $2^{\mathcal{L}_Y^\infty}$ and 2^Y be the hyperspaces of nonempty, closed (and hence compact) subsets of $\mathcal{L}_Y^\infty$ and Y respectively, equipped with the Hausdorf metrics $h_{\rho_{\mathcal{L}_Y^\infty}}$ and h_{ρ_Y} induced by the M-convex metrics, $\rho_{\mathcal{L}_Y^\infty}$ and ρ_Y. Because the metric spaces, $(\mathcal{L}_Y^\infty, \rho_{\mathcal{L}_Y^\infty})$ and (Y, ρ_Y) are compact and M-convex, the hyperspaces, $(2^{\mathcal{L}_Y^\infty}, h_{\rho_{\mathcal{L}_Y^\infty}})$ and $(2^Y, h_{\rho_Y})$, are compact and M-convex (see Duda, 1970). We will refer to the summary of primitives above as [FPP].

Note that due to compactness, the uc correspondence, $\mathcal{P}(\cdot, \cdot)$, has a measurable graph - meaning that

$$\begin{aligned} Gr\mathcal{P}(\cdot, \cdot) \quad &: \quad = \{(\omega, v, y) \in \Omega \times \mathcal{L}_Y^\infty \times Y : y \in \mathcal{P}(\omega, v)\} \\ &\in \quad B_\Omega \times B_{\mathcal{L}_Y^\infty} \times B_Y. \end{aligned}$$

We will denote by $\mathcal{P}^{USCO} := \{\mathcal{P}(\omega, \cdot) : \omega \in \Omega\}$, the upper semicontinuous part (i.e., USCO part) of $\mathcal{P}(\cdot, \cdot)$, and by $\mathcal{P}^{B_\Omega} := \{\mathcal{P}(\cdot, v) : v \in \mathcal{L}_Y^\infty\}$, the measurable part of $\mathcal{P}(\cdot, \cdot)$, with graphs,

$$\left.\begin{array}{c} Gr\mathcal{P}_\omega := \{(v, y) \in \mathcal{L}_Y^\infty \times Y : y \in \mathcal{P}(\omega, v)\}, \\ Gr\mathcal{P}_v := \{(\omega, y) \in \Omega \times Y : y \in \mathcal{P}(\omega, v)\}. \end{array}\right\}$$

3 USCOs, Upper Caratheodory Correspondences, and Approximate Caratheodory Selections

3.1 USCOs

For compact metric spaces $(\mathcal{L}_Y^\infty, \rho_{\mathcal{L}_Y^\infty})$ and (Y, ρ_Y), let $\mathcal{U}_{\mathcal{L}_Y^\infty - Y} := \mathcal{U}(\mathcal{L}_Y^\infty, 2^Y)$ denote the collection of all upper semicontinuous correspondences taking nonempty, ρ_Y-closed (and hence ρ_Y-compact) values in Y. Following the literature, we will call such mappings, USCOs (see Crannell, Franz, and LeMasurier, 2005, Anguelov and Kalenda, 2009, and Hola and Holy, 2009). Given any USCO, $\Psi \in \mathcal{U}_{\mathcal{L}_Y^\infty - Y}$, denote by $\mathcal{U}_{\mathcal{L}_Y^\infty - Y}[\Psi]$ the collection of all sub-USCOs belonging to Ψ, that is, all USCOs $\phi \in \mathcal{U}_{\mathcal{L}_Y^\infty - Y}$ whose graph,

$$Gr\phi := \{(v, y) \in \mathcal{L}_Y^\infty \times Y : y \in \phi(v)\},$$

is contained in the graph of Ψ,

$$Gr\Psi := \{(v, y) \in \mathcal{L}_Y^\infty \times Y : y \in \Psi(v)\}.$$

We will call any sub-USCO, $\phi \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}Y}[\Psi]$ a minimal USCO belonging to Ψ, if for any other sub-USCO, $\psi \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}Y}[\Psi]$, $Gr\psi \subseteq Gr\phi$ implies that $Gr\psi = Gr\phi$. We will use the special notation, $[\Psi]$, to denote the collection of all minimal USCOs belonging to Ψ. We know that for any USCO Ψ, $[\Psi] \neq \varnothing$ (see Drewnowski and Labuda, 1990). In general, we say that ψ is a minimal USCO, if for any other USCO $\phi \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}Y}$, $Gr\phi \subseteq Gr\psi$ implies that $Gr\phi = Gr\psi$. Let $\mathcal{M}_{\mathcal{L}_Y^\infty\text{-}Y}$ denote the collection of all minimal USCOs.

Finally, we say that an USCO, $\Psi \in \mathcal{U}_{\mathcal{L}_Y^\infty - Y}$, is quasi-minimal if for some $\psi \in \mathcal{U}_{\mathcal{L}_Y^\infty - Y}$, $[\Psi] = \{\psi\}$ (i.e., Ψ has one and only one minimal USCO). Let $\mathcal{Q}_{\mathcal{L}_Y^\infty - Y}$ denote the collection of all quasi-minimal USCOs. We will denote by

$$S_\Psi := \{v \in \mathcal{L}_Y^\infty : \Psi(v) \text{ is a singleton}\},$$

the subset where Ψ takes singleton values. Under our primitives and assumptions, if $\Psi \in \mathcal{Q}_{\mathcal{L}_Y^\infty - Y}$, then by Lemma 7 in Anguelov and Kalenda (2009), S_Ψ is a residual set - and in particular, a G_δ set $\rho_{\mathcal{L}_Y^\infty}$-dense in $\mathcal{L}_Y^\infty$.

3.2 Minimal Upper Caratheodory Correspondences and Minimal USCOs

Let $\mathcal{UC}_{\Omega \times \mathcal{L}_Y^\infty\text{-}Y}$ denote the collection of all upper Charatheodory mapping defined on $\Omega \times \mathcal{L}_Y^\infty$ taking nonempty, ρ_Y-closed (and hence ρ_Y-compact) values in Y. For $\mathcal{P}(\cdot, \cdot) \in \mathcal{UC}_{\Omega \times \mathcal{L}_Y^\infty\text{-}Y}$, let

$$\mathcal{UC}^{\mathcal{P}} := \mathcal{UC}_{\Omega \times \mathcal{L}_Y^\infty\text{-}Y}[\mathcal{P}(\cdot, \cdot)]$$

denote the collection of all upper Caratheodory mappings, $p(\cdot, \cdot)$, belonging to $\mathcal{P}(\cdot, \cdot)$. Thus, $p(\cdot, \cdot) \in \mathcal{UC}^{\mathcal{P}}$ if and only if $p(\cdot, \cdot) \in \mathcal{UC}_{\Omega \times \mathcal{L}_Y^\infty\text{-}Y}$ and

$$Grp(\omega, \cdot) \subset Gr\mathcal{P}(\omega, \cdot) \text{ for all } \omega.$$

We will be interested in sub-uc-correspondences, $p(\cdot, \cdot) \in \mathcal{UC}^{\mathcal{P}}$, with the property that for each ω, $p(\omega, \cdot)$ is a *minimal USCO belonging to* $\mathcal{P}(\omega, \cdot)$. Already we know that (i) for each ω, $[\mathcal{P}(\omega, \cdot)]$ is nonempty (e.g., see Drewnowski and Labuda, 1990), and (ii) for each ω and $p(\omega, \cdot) \in [\mathcal{P}(\omega, \cdot)]$, $S_{p(\omega, \cdot)}$ is a G_δ set $\rho_{\mathcal{L}_Y^\infty}$-dense in $\mathcal{L}_Y^\infty$, and therefore, for each $(\omega, v) \in \Omega \times S_{\eta(\omega, \cdot)}$, $p(\omega, v)$ is single-valued. What we don't know is whether or not the uc correspondence, $\mathcal{P}(\cdot, \cdot)$, contains a sub-uc-correspondence, $p(\cdot, \cdot)$, such that for each ω, $p(\omega, \cdot)$ is a minimal USCO belonging to $\mathcal{P}(\omega, \cdot)$. We call any such sub-uc correspondence a minimal uc correspondence, and we denote by,

$$\mathcal{MUC}^{\mathcal{P}} := \left\{p(\cdot, \cdot) \in \mathcal{UC}^{\mathcal{P}} : p(\omega, \cdot) \in [\mathcal{P}(\omega, \cdot)] \text{ for all } \omega\right\},$$

the collection of minimal uc correspondences belonging to $\mathcal{P}(\cdot, \cdot) \in \mathcal{UC}_{\Omega \times \mathcal{L}_Y^\infty\text{-}Y}$.

Our first result shows that for any $\mathcal{P} \in \mathcal{UC}_{\Omega \times \mathcal{L}_Y^\infty\text{-}Y}$, $\mathcal{MUC}^{\mathcal{P}}$ is nonempty.

Theorem 1 (Nonemptiness of $MUC^{\mathcal{P}}$) *Suppose assumptions [FPP] hold. For any uc correspondence, $P(\cdot, \cdot) \in UC_{\Omega \times \mathcal{L}_Y^\infty\text{-}Y}$, there exists a uc correspondence, $p(\cdot, \cdot) \in UC_{\Omega \times \mathcal{L}_Y^\infty\text{-}Y}$, such that $p(\omega, \cdot) \in [P(\omega, \cdot)]$ for all ω.*

3.3 ε-Caratheodory Selections for Minimal UC Correspondences

We begin with the basic definitions.

Definition 1 (Caratheodory Functions)
Given $\mathcal{P}(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times\mathcal{L}^{\infty}_{Y}\text{-}Y}$, let $p(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{P}}$ (i.e., for each ω, $p(\omega,\cdot) \in [\mathcal{P}(\omega,\cdot)]$). A function,

$$g(\cdot,\cdot) : \Omega \times \mathcal{L}^{\infty}_{Y} \longrightarrow Y,$$

is said to be Caratheodory if (i) for each $\omega \in \Omega$ the function, $g(\omega,\cdot) := g_{\omega}(\cdot)$, defined on $\mathcal{L}^{\infty}_{Y}$ with values in Y is $\rho_{\mathcal{L}^{\infty}_{Y}}$-$\rho_Y$-continuous, and if (ii) for each $v \in \mathcal{L}^{\infty}_{Y}$ the function, $g(\cdot,v) := g_v(\cdot)$ is (B_{Ω}, B_Y)-measurable.

Definition 2 (ε-Caratheodory Selections of Minimal UC Correspondences)
Given $\mathcal{P}(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times\mathcal{L}^{\infty}_{Y}\text{-}Y}$, let $p(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{P}}$ (i.e., for each ω, $p(\omega,\cdot) \in [\mathcal{P}(\omega,\cdot)]$). A function,

$$g^{\varepsilon}(\cdot,\cdot) : \Omega \times \mathcal{L}^{\infty}_{Y} \longrightarrow Y,$$

is said to be ε-Caratheodory Selection of $p(\cdot,\cdot)$ if for each $(\omega,v) \in \Omega \times \mathcal{L}^{\infty}_{Y}$ and each $(v, g^{\varepsilon}(\omega,v)) \in \mathcal{L}^{\infty}_{Y} \times Y$ there exists $(\overline{v},\overline{y}) \in Grp(\omega,\cdot)$ such that

$$\rho_{\mathcal{L}^{\infty}_{Y}}(v,\overline{v}) + \rho_Y(g^{\varepsilon}(\omega,v),\overline{y}) < \varepsilon,$$

or equivalently, such that the Caratheodory function, $g^{\varepsilon} : \Omega \times \mathcal{L}^{\infty}_{Y} \longrightarrow Y$, has the property that for each ω

$$Grg^{\varepsilon}(\omega,\cdot) \subset B_{\rho_{\mathcal{L}^{\infty}_{Y}\times Y}}(\varepsilon, Grp(\omega,\cdot)).$$

We say that the minimal uc correspondence, $p(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{P}}$, is Caratheodory approximable if for each $\varepsilon > 0$, $p(\cdot,\cdot)$ has an ε-Caratheodory Selection.

By Theorem 4.2 in Kucia and Nowak (2000), a sufficient condition for $p(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{P}}$ to have for each $\varepsilon > 0$ an ε-Caratheodory selection is for each ω the minimal USCO, $p(\omega,\cdot)$, to have an ε-continuous selection.

Definition 3 (ε-Continuous Selections of Minimal USCOs, $p(\omega,\cdot)$)
Given $\mathcal{P}(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times\mathcal{L}^{\infty}_{Y}\text{-}Y}$, let $p(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{P}}$ (i.e., for each ω, $p(\omega,\cdot) \in [\mathcal{P}(\omega,\cdot)]$). A function,

$$f^{\varepsilon}(\cdot) : \mathcal{L}^{\infty}_{Y} \longrightarrow Y,$$

is said to be ε-continuous selection of $p(\omega,\cdot)$ if for each $v \in \mathcal{L}^{\infty}_{Y}$ and each $(v, f^{\varepsilon}(v)) \in \mathcal{L}^{\infty}_{Y} \times Y$ there exists $(\overline{v},\overline{y}) \in Grp(\omega,\cdot)$ such that

$$\rho_{\mathcal{L}^{\infty}_{Y}}(v,\overline{v}) + \rho_Y(f^{\varepsilon}(v),\overline{y}) < \varepsilon,$$

or equivalently, such that the continuous function, $f^{\varepsilon} : \mathcal{L}^{\infty}_{Y} \longrightarrow Y$, has the property that

$$Grf^{\varepsilon}(\cdot) \subset B_{\rho_{\mathcal{L}^{\infty}_{Y}\times Y}}(\varepsilon, Grp(\omega,\cdot))$$

We say that the minimal uc correspondence, $p(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{P}}$, is approximable if for each ω and for each $\varepsilon > 0$, the minimal USCO, $p(\omega,\cdot)$, has an ε-continuous selection.

Given assumptions [FPP], it follows from Theorem 5.12 in Gorneiwicz, Granas, and Kryszewski (1991) that if $p(\omega,\cdot)$ is R_{δ}-valued, then for any $\varepsilon > 0$, $p(\omega,\cdot)$ has an ε-continuous selection. One of the main objectives of the paper is to identify conditions the correspondence $p(\omega,\cdot)$ must satisfy in order to guarantee that $p(\omega,\cdot)$ is R_{δ}-valued. Under the assumptions [FPP], we will show that if $p(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{P}}$ is $3M$ at (ω,v) for all $(\omega,v) \in \Omega\times\mathcal{L}^{\infty}_{Y}$, then $p(\cdot,\cdot)$ takes minimally essential continuum values (and conversely). Moreover, because these continuum values are contained in Y, a globally arc-smooth continuum (i.e., a continuum that is arc-smooth at each of its points), $p(\cdot,\cdot)$ takes R_{δ}-values.

4 Essential Sets, the $3M$ Property, and Continuum Values

We begin with the definitions of essential and minimally essential values belonging to an upper Caratheodory correspondence's USCO part for a particular ω - where essential and minimally essential are in the sense of Fort (1950), Kinoshita (1952), and Jiang (1962). We also reintroduce the $3M$ property of Page and Resende (2011). We then show that, under assumptions [FPP], all $3M$ minimal uc correspondences are minimally essentially valued and if they have the $3M$ property, they are continuum valued, and conversely, if they are continuum valued, they are $3M$.

4.1 Essential Sets and the $3M$ Property

Let $\mathcal{P}(\cdot,\cdot)$ be an upper Caratheodory correspondence with USCO part,

$$\mathcal{P}^{USCO} = \left\{\mathcal{P}_{\omega}(\cdot) \in \mathcal{U}_{\mathcal{L}^{\infty}_{Y}\text{-}Y} : \omega \in \Omega\right\}.$$

Definition 4 (Essential Sets)
(1) (Essential Set) A nonempty closed subset $e_{\omega}(v^0)$ of $\mathcal{P}(\omega,v^0)$ is said to be essential for $\mathcal{P}_{\omega}(\cdot)$ at $v^0 \in \mathcal{L}^{\infty}_{Y}$ if for each $\varepsilon > 0$ there exists $\delta^{\varepsilon} > 0$ such that for all $v \in B_{\rho_{\mathcal{L}^{\infty}_{Y}}}(\delta^{\varepsilon}, v^0)$,

$$\mathcal{P}(\omega,v) \cap B_{\rho_Y}(\varepsilon, e_{\omega}(v^0)) \neq \emptyset.$$

We will denote by $\mathcal{E}[\mathcal{P}(\omega,v^0)] \subset 2^{\mathcal{P}(\omega,v^0)}$ the collection of all nonempty, ρ_Y-closed subsets of $\mathcal{P}(\omega,v^0)$ essential for $\mathcal{P}_{\omega}(\cdot)$ at $v^0 \in \mathcal{L}^{\infty}_{Y}$.
(2) (Minimal Essential Set) A nonempty closed subset $m_{\omega}(v^0)$ of $\mathcal{P}(\omega,v^0)$ is said to be minimally essential for $\mathcal{P}_{\omega}(\cdot)$ at $v^0 \in \mathcal{L}^{\infty}_{Y}$ if (i) $m_{\omega}(v^0) \in \mathcal{E}[\mathcal{P}(\omega,v^0)]$ and if

(ii) $m_\omega(v^0)$ is a minimal element of $\mathcal{E}[\mathcal{P}(\omega, v^0)]$ ordered by set inclusion (i.e., if $e_\omega(v^0) \in \mathcal{E}[\mathcal{P}(\omega, v^0)]$ and $e_\omega(v^0) \subseteq m_\omega(v^0)$ then $e_\omega(v^0) = m_\omega(v^0)$). We will denote by $\mathcal{E}^*[\mathcal{P}(\omega, v^0)]$ the collection of all nonempty, closed subsets of $\mathcal{P}(\omega, v^0)$ minimally essential for $\mathcal{P}_\omega(\cdot)$ at $v^0 \in \mathcal{L}^\infty_Y$.

The $3M$ property (i.e., the 3 misses property) is defined as follows:

Definition 5 (The $3M$ Property)
Let $\mathcal{P}(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times\mathcal{L}^\infty_Y\text{-}Y}$ be a uc correspondence and consider the minimal uc correspondence, $p(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{P}}$. We say that $p(\cdot,\cdot)$ is 3M at (ω^0, v^0) if the minimal ω^0-USCO, p_{ω^0}, has the property at v^0 that for any $\delta > 0$ and for any pair of nonempty disjoint closed sets, F^1 and F^2, in Y there exists v^1 and v^2 in $B_{\rho_{\mathcal{L}^\infty_Y}}(\delta, v^0)$ such that

$$p_{\omega^0}(v^1) \cap F^1 = \emptyset \text{ and } p_{\omega^0}(v^2) \cap F^2 = \emptyset,$$

(i.e., $p_{\omega^0}(\cdot)$ misses F^1 at v^1 and misses F^2 at v^2 for some v^1 and v^2 in $B_{\rho_{\mathcal{L}^\infty_Y}}(\delta, v^0)$), then there exists a third point, v^3, in the larger open ball, $B_{\rho_{\mathcal{L}^\infty_Y}}(3\delta, v^0)$, such that

$$p_{\omega^0}(v^3) \cap [F^1 \cup F^2] = \emptyset,$$

(i.e., $p_{\omega^0}(\cdot)$ misses $F^1 \cup F^2$ at $v^3 \in B_{\rho_{\mathcal{L}^\infty_Y}}(3\delta, v^0)$). We say that a uc correspondence, $\mathcal{P}(\cdot,\cdot)$, is $3M$ if for some $p(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{P}}$, $p(\cdot,\cdot)$ is $3M$ at (ω, v) for all $(\omega, v) \in \Omega \times \mathcal{L}^\infty_Y$. We will denote by $\mathcal{MUC}^{\mathcal{P}}_{3M}$ the collection of all $3M$ minimal uc correspondences belonging to $\mathcal{P}(\cdot,\cdot)$, and we will denote by $\mathcal{UC}^{3M}_{\Omega\times\mathcal{L}^\infty_Y\text{-}Y}$, the collection of all uc correspondences such that $\mathcal{MUC}^{\mathcal{P}}_{3M} \neq \varnothing$.

We say that a uc correspondence, $\mathcal{P}(\cdot,\cdot)$, is quasi-minimal at ω, if the ω-USCO, $\mathcal{P}_\omega(\cdot) := \mathcal{P}(\omega,\cdot)$ is quasi-minimal. We say that $\mathcal{P}(\cdot,\cdot)$ is quasi-minimal if the ω-USCO, $\mathcal{P}_\omega(\cdot) := \mathcal{P}(\omega,\cdot)$ is quasi-minimal for all ω. We will denote the collection of quasi-minimal upper Caratheodory correspondences by $\mathcal{Q}_{\Omega\times\mathcal{L}^\infty_Y-Y}$. The following Theorem establishes a fundamental fact about quasi-minimal uc correspondences: any minimal uc correspondence belonging to a quasi-minimal uc correspondence takes minimally essential values.

Theorem 2 (The Quasi-Minimal Theorem for UC Correspondences)
Suppose assumptions [FPP] hold. Let $\mathcal{P}(\cdot,\cdot)$ be a quasi-minimal uc correspondence. Then, for $p(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{P}}$, $p(\omega, v) \in \mathcal{E}^*[\mathcal{P}(\omega, v)]$ for each $(\omega, v) \in \Omega \times \mathcal{L}^\infty_Y$.

If $\mathcal{P}(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times\mathcal{L}^\infty_Y\text{-}Y}$ is quasi-minimal, then it can contain only one sub uc correspondence, namely, its minimal uc correspondence.

Theorem 3 (Quasi-Minimal USCOs and Minimal USCOs)
Suppose assumptions [FPP] hold. Let $\mathcal{P}(\cdot,\cdot)$ be a quasi-minimal uc correspondence. If $p(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{P}}$, then for any $\phi \in \mathcal{U}_{\rho_{\mathcal{L}^\infty_Y}\text{-}\rho_Y}$ with $Gr\phi$ is a proper subset of $Gr\mathcal{P}_{\omega^0}$ for some ω^0, $Gr\phi = Grp_{\omega^0}$.

Let $\mathcal{MUC}^{\mathcal{P}}_{3M}$ denote the collection of minimal uc correspondences, $p(\cdot,\cdot)$, belonging to $\mathcal{P} \in \mathcal{UC}_{\Omega\times\mathcal{L}^\infty_Y\text{-}Y}$ such that for each $\omega \in \Omega$, $p(\omega,\cdot)$, an USCO defined on $\mathcal{L}^\infty_Y$ taking nonempty closed values in Y, is $3M$. Let $\mathcal{MUC}^{\mathcal{P}}_{C(Y)}$ denote the collection of minimal uc correspondences, $p(\cdot,\cdot)$, belonging to $\mathcal{P} \in \mathcal{UC}_{\Omega\times\mathcal{L}^\infty_Y\text{-}Y}$ such that for each $\omega \in \Omega$, $p(\omega,\cdot)$, an USCO defined on $\mathcal{L}^\infty_Y$ taking nonempty *closed and connected* values in Y - i.e., values in $C(Y)$, where $C(Y)$ denotes the hyperspace of nonempty closed and connected subsets of Y (i.e., subcontinua of Y). Our next Theorem establishes that for any minimal uc correspondence, $p(\cdot,\cdot)$, defined on an M-convex, compact metric space with values in 2^Y, $p(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{P}}_{3M}$ if and only if $p(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{P}}_{C(Y)}$.

Theorem 4 (For $P(\cdot,\cdot) \in UC_{\Omega\times\mathcal{L}^\infty_Y\text{-}Y}$, $MUC^{\mathcal{P}}_{3M} = MUC^{\mathcal{P}}_{C(Y)}$)
Let (Ω, B_Ω) be a measurable space of states with Borel σ-field B_Ω and let $(\mathcal{L}^\infty_Y, \rho_{\mathcal{L}^\infty_Y})$ and (Y, ρ_Y) be Peano continua with M-convex metrics $\rho_{\mathcal{L}^\infty_Y}$ and ρ_Y. Given any $\mathcal{P}(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times\mathcal{L}^\infty_Y\text{-}Y}$, any $p(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{P}}$, and any state-parameter pair, $(\omega, v) \in \Omega \times \mathcal{L}^\infty_Y$, the following statements are equivalent:

$$\left.\begin{array}{c} (1)\ p_\omega(\cdot) \text{ is } 3M \text{ at } v. \\[1em] (2)\ p_\omega(v) \in \mathcal{E}^*[p_\omega(v)] \cap C(Y). \end{array}\right\}$$

4.2 $3M$ Minimal Upper Caratheodory Correspondences Are R_δ-Valued

First, recall the formal definition of an R_δ set.

Definition 6 (R_δ-Sets)
A set M is called an R_δ-set, denoted $M \in R_\delta$, if there exists a sequence of compact, nonempty contractible sets, $\{M^n\}_n$ such that

$$M^{n+1} \subseteq M^n \text{ for every } n \\ \text{and} \\ M = \cap_{n=1}^\infty M^n.$$

A key fact allowing us to show that $3M$ minimal uc correspondences, $p(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{P}}$, are R_δ-valued is that each such minimal uc correspondences is continuum-valued and has minimally essential values contained in a *globally arc-smooth continuum*, $Y \subset R^m$ (see Section

25.8 in Illanes and Nadler, 1999).[1] By Lemma 19.1 in Illanes and Nadler (1999), each subcontinuum, $p(\omega, v)$, contained in Y is given by the intersection of a decreasing sequence of Peano continua, $\{B^n\}_n$, contained in Y, and therefore, the intersection of a decreasing sequence of M-convex continua contained in Y. Thus, if we can show that each Peano continua, $B^n \subset Y$, making up the decreasing sequence of Peano continua, $\{B^n\}_n$, such that $p(\omega, v) = \cap_n B^n$, is contractible, then by the definition of R_δ-valuedness, $p(\omega, v)$ is an R_δ set. We have the following result:

Theorem 5 (The R_δ Theorem: for each (ω, v), $p(\omega, v)$ is an R_δ-set)
Suppose assumptions [FPP] hold. Let $\mathcal{P}(\cdot,\cdot)$ be a uc correspondence. If $p(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{P}}_{3M}$, then for each $(\omega, v), \in \Omega \times \mathcal{L}^\infty_Y$, $p(\omega, v)$ is an R_δ-set.

5 Caratheodory Approximable $3M$ Minimal UC Correspondences

Our main result on $3M$ minimal uc correspondences and Caratheodory approximation is the following:

Theorem 6 (Sufficient Conditions for Caratheodory Approximability)
Suppose assumptions [FPP] hold. Let $\mathcal{P}(\cdot,\cdot)$ be a $3M$ uc correspondence. If $p(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{P}}_{3M}$, then $p(\cdot,\cdot)$ is Caratheodory approximable.

Proof: In light of Theorem 4.2 in Kucia and Nowak, (2000), it suffices to show that for each $\omega \in \Omega$, the minimal USCO $p(\omega, \cdot)$ is approximable. By Theorem above, we have that for each ω, the minimal USCO, $p(\omega, \cdot) \in [\mathcal{P}_\omega(\cdot)]$ is R_δ-valued. By Corollary 5.6 in Gorniewicz, Granas, and Kryszewski (1991), because $p(\omega, \cdot)$ is a mapping defined on the ANR (absolute neighborhood retract) space $\mathcal{L}^\infty_Y$ taking nonempty, compact, R_δ values in the ANR space Y, the minimal USCO, $p(\omega, \cdot)$, is ∞-proximally connected valued, and therefore a J mapping. Thus, by Theorem 5.12 in Gorniewicz, Granas, and Kryszewski (1991), $p(\omega, \cdot)$ is approximable. Because this is true for each $\omega \in \Omega$, by Theorem 4.2 in Kucia and Nowak (2000), $p(\cdot,\cdot)$ is Caratheodory approximable. **Q.E.D.**

If, for example, $\mathcal{P}(\cdot,\cdot)$ is such that for each ω, $\mathcal{P}(\omega, \cdot)$ contains a convex-valued or star-shape valued minimal USCO, $p(\omega, \cdot)$ - and thus, an R_δ-valued minimal USCO - then $\mathcal{P}(\cdot,\cdot)$ is approximable. In addition, if for each ω, $\mathcal{P}(\omega, \cdot)$ contains a minimal USCO, $p(\omega, \cdot)$, taking arc-like continuum values, arc-smooth continuum values, or dendritic values, then $p(\omega, \cdot)$ is contractibly-valued - and thus, is R_δ-valued (for related results see Cellina, 1969 and Beer, 1983, 1988 and 1993) - implying, via Kucia and Nowak (2000) that $\mathcal{P}(\cdot,\cdot)$ is Caratheodory approximable.

[1] *A continuum is said to be globally arc smooth if at each point,* $y \in Y$, there is a continuous function, $A_y(\cdot)$, specifying an arc structure in Y.

6 Selections and Fixed Points of $3M$ UC Correspondences

Many, if not all, parameterized, state-contingent m-person noncooperative games have globally arc-smooth payoff spaces. Our main selection result for $3M$ uc correspondences mapping into globally arc-smooth continua is the following:

Theorem 7 (A Selection Result for $3M$ UC Correspondences Mapping into Globally Arc-Smooth Continua)
Suppose assumptions [FPP] hold. Let

$$\mathcal{P}(\cdot,\cdot) : \Omega \times \mathcal{L}^\infty_Y \longrightarrow 2^Y$$

be a uc correspondence taking nonempty closed values in the everywhere arc-smooth continuum

$$Y := [-M, M]^m \subset R^m.$$

If $\mathcal{MUC}^{\mathcal{P}}_{3M} \neq \varnothing$, then there exists $v^* \in \mathcal{L}^\infty_Y$ such that

$$v^*(\omega) \in \mathcal{P}(\omega, v^*) \ \textit{a.e.}\ [\mu].$$

An immediate Corollary of Theorem 7 is the following fixed point result.

Theorem 8 (Fixed Points for Measurable-Selection-Valued Correspondences Induced by $3M$ Caratheodory Compositions)
Let

$$\begin{array}{rcl} (\omega, v) & \longrightarrow & \mathcal{P}(\omega, v) \\ & : & = \{U \in Y : U = U(\omega, v, x) \ \textit{for some}\ x \in \mathcal{N}(\omega, v)\} \\ & : & = U(\omega, v, \mathcal{N}(\omega, v)) \end{array}$$

be a Caratheodory composition. If $\mathcal{P}(\cdot,\cdot)$ is $3M$, then $v \longrightarrow \mathcal{S}^\infty(\mathcal{P}_v)$ has fixed points (i.e., there exists $v^* \in \mathcal{L}^\infty_Y$ such that $v^* \in \mathcal{S}^\infty(\mathcal{P}_{v^*})$).

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