1 Introduction

We establish a new fixed point result for measurable-selection-valued correspondences with nonconvex and possibly disconnected values induced by upper Caratheodory correspondences (i.e., a uc correspondences). We show that if the upper semicontinuous part of a uc correspondence is approximable (via continuous functions), then the induced measurable-selection-valued correspondence has fixed points in its set of measurable selections. Also, we show that if the uc correspondence has, in each state, an upper semicontinuous part containing a Rδ-valued (equivalently, contractible-valued or dendritically-valued) upper semicontinuous sub-correspondence, then the upper semicontinuous part is state-by-state, approximable - implying that the uc correspondence is Caratheodory approximable (via Caratheodory functions). Are there uc correspondences having such properties? We show that if the underlying uc correspondence has the 3M property - a property introduced in Page and Resende (2011) - then it will contain a minimal uc correspondence having the property that in each state the upper semicontinuous part of this minimal uc correspondence is an Rδ-valued minimal USCO.

An excellent example of a 3M upper Caratheodory correspondence is the Nash payoff correspondence belonging to any m-person, noncooperative parameterized state-contingent game (i.e., belonging to any PSΣ). Under mild conditions on primitives, Page and Fu (2018) have shown that the Nash payoff correspondence belonging to any PSΣ is 3M, and as a consequence, any PSΣ has an approximable Nash payoff correspondence - and therefore, has a Nash payoff selection correspondence having fixed points.

2 Primitives of the Fixed Point Problem

To begin, suppose (Ω, BΩ, μ) is a probability space of states, ω, with state space, Ω, a complete separable metric space, equipped with Borel σ-field, BΩ, and probability measure, μ. Let D := {1, 2, 3, …, m} be a finite index set and for each d ∈ D, let $Y_d := [-M, M] \subset \mathbb{R}$ be the closed bounded interval. For each d ∈ D, the interval $Y_d$ contains the range of the functions in the function space, $\mathcal{L}_Y$. Here, $\mathcal{L}_Y$ consists of all μ-equivalence classes of real-valued measurable functions, $v_d$, defined on Ω with values in $Y_d$ a.e. [μ]. Because $\mathcal{L}_Y$ is weak star compact and metrizable as well as convex and locally convex (and therefore connected and locally connected - and therefore a Peano continuum), $\mathcal{L}_Y$ can be equipped with an M-convex metric $\rho_{\mathcal{L}_Y}$ compatible with the weak star topology inherited from $\mathcal{L}_Y$ (the Banach space of all essentially bounded, real-valued measurable functions). The weak star topology in $\mathcal{L}_Y$ is denoted by $w^*$ or by $\sigma(\mathcal{L}_Y, B_{\mathcal{L}_Y})$. Now let

$$\mathcal{L}_Y := \mathcal{L}_{Y_1} \times \cdots \times \mathcal{L}_{Y_m},$$

where $Y := Y_1 \times \cdots \times Y_m$. Equip $\mathcal{L}_Y$ with the M-convex sum metric

$$\rho_{\mathcal{L}_Y} := \sum_{d=1}^m \rho_{\mathcal{L}_{Y_d}}.$$

Note that $(\mathcal{L}_Y, \rho_{\mathcal{L}_Y})$ is an M-convex compact metric space of $Y$-valued measurable functions, and therefore, $\mathcal{L}_Y$ has a Borel σ-field, $B_{\mathcal{L}_Y}$, generated by its $\rho_{\mathcal{L}_Y}$-open sets.

Consider the upper Caratheodory correspondence,

$$\mathcal{P} : \Omega \times \mathcal{L}_Y \rightarrow 2^Y,$$

jointly measurable in $(\omega, v) \in \Omega \times \mathcal{L}_Y$ and upper semicontinuous in $v$ for each $\omega$, taking nonempty, $\rho_Y$-closed (and hence, $\rho_Y$-compact) values in $Y$, a closed, bounded convex subset of $\mathbb{R}^m$. Equip $Y$ with the sum metric,

$$\rho_Y := \sum_{d=1}^m \rho_{Y_d}.$$
where for each $d$, $\rho_d$ is the absolute value metric on $[-M, M]$. Finally, let $2^\mathbb{C}_Y$ and $2^Y$ be the hyperspaces of nonempty, closed (and hence compact) subsets of $\mathcal{L}_Y$ and $Y$ respectively, equipped with the Hausdorff metrics $h_{\mathcal{L}_Y}$ and $h_Y$ induced by the $M$-convex metrics, $\rho_{\mathcal{L}_Y}$ and $\rho_Y$. Because the metric spaces, $(\mathcal{L}_Y, \rho_{\mathcal{L}_Y})$ and $(Y, \rho_Y)$ are compact and $M$-convex, the hyperspaces, $(2^\mathbb{C}_Y, h_{\mathcal{L}_Y})$ and $(2^Y, h_Y)$, are compact and $M$-convex (see Duda, 1970). We will refer to the summary of primitives above as $\text{[FPP]}$.

Note that due to compactness, the uc correspondence, $\mathcal{P}(\cdot, \cdot)$, has a measurable graph - meaning that

$$G\mathcal{P}(\cdot, \cdot) := \{ (\omega, v, y) \in \Omega \times 2^\mathbb{C}_Y \times Y : y \in \mathcal{P}(\omega, v) \} \in B_\Omega \times B_{2^\mathbb{C}_Y} \times B_Y.$$  

We will denote by $\mathcal{P}^\text{USCO} := \{ \mathcal{P}(\cdot, \cdot) : \omega \in \Omega \}$, the upper semicontinuous part (i.e., USCO part) of $\mathcal{P}(\cdot, \cdot)$, and by $\mathcal{P}^\text{B_0} := \{ \mathcal{P}(\cdot, v) : v \in \mathbb{C}_Y \}$, the measurable part of $\mathcal{P}(\cdot, \cdot)$, with graphs,

$$G\mathcal{P}_\omega := \{ (v, y) \in \mathbb{C}_Y \times Y : y \in \mathcal{P}(\omega, v) \},$$  

$$G\mathcal{P}_v := \{ (\omega, y) \in \Omega \times Y : y \in \mathcal{P}(\omega, v) \}.$$  

### 3 USCOs, Upper Caratheodory Correspondences, and Approximate Caratheodory Selections

#### 3.1 USCOs

For compact metric spaces $(\mathcal{L}_Y, \rho_{\mathcal{L}_Y})$ and $(Y, \rho_Y)$, let $\mathcal{U}_{\mathcal{L}_Y-Y} := \mathcal{U}(\mathcal{L}_Y, 2^Y)$ denote the collection of all upper semicontinuous correspondences taking nonempty, $\rho_Y$-closed (and hence $\rho_Y$-compact) values in $Y$. Following the literature, we will call such mappings, USCOs (see Crannell, Franz, and LeMasurier, 2005, Anguelov and Kalenda, 2009, and Hola and Holy, 2009). Given any USCO, $\Psi \in \mathcal{U}_{\mathcal{L}_Y-Y}$, denote by $\mathcal{U}_{\mathcal{L}_Y-Y}[\Psi]$ the collection of all sub-USCOs belonging to $\Psi$, that is, all USCOs $\phi \in \mathcal{U}_{\mathcal{L}_Y-Y}$ whose graph,

$$G\phi := \{ (v, y) \in \mathcal{L}_Y \times Y : y \in \phi(v) \},$$  

is contained in the graph of $\Psi$,

$$G\Psi := \{ (v, y) \in \mathcal{L}_Y \times Y : y \in \Psi(v) \}.$$  

We will call any sub-USCO, $\phi \in \mathcal{U}_{\mathcal{L}_Y-Y}[\Psi]$ a minimal USCO belonging to $\Psi$, if for any other sub-USCO, $\psi \in \mathcal{U}_{\mathcal{L}_Y-Y}[\Psi]$, $G\phi \subseteq G\psi$ implies that $G\phi = G\psi$. We will use the special notation, $[\Psi]$, to denote the collection of all minimal USCOs belonging to $\Psi$. We know that for any USCO $\Psi$, $[\Psi] \neq \emptyset$ (see Drewnowski and Labuda, 1990). In general, we say that $\psi$ is a minimal USCO, if for any other USCO $\phi \in \mathcal{U}_{\mathcal{L}_Y-Y}$, $G\phi \subseteq G\psi$ implies that $G\phi = G\psi$. Let $\mathcal{M}_{\mathcal{L}_Y-Y}$ denote the collection of all minimal USCOs.

Finally, we say that an USCO, $\Psi \in \mathcal{U}_{\mathcal{L}_Y-Y}$, is quasi-minimal if for some $\psi \in \mathcal{U}_{\mathcal{L}_Y-Y}$, $[\psi] = \{ \psi \}$ (i.e., $\Psi$ has one and only one minimal USCO). Let $\mathcal{Q}_{\mathcal{L}_Y-Y}$ denote the collection of all quasi-minimal USCOs. We will denote by

$$S_\Psi := \{ v \in \mathbb{C}_Y : \Psi(v) \text{ is a singleton} \},$$  

the subset where $\Psi$ takes singleton values. Under our primitives and assumptions, if $\Psi \in \mathcal{Q}_{\mathcal{L}_Y-Y}$, then by Lemma 7 in Anguelov and Kalenda (2009), $S_\Psi$ is a residual set - and in particular, a $G_\delta$ set $\rho_{\mathcal{L}_Y}$-dense in $\mathbb{C}_Y$.

#### 3.2 Minimal Upper Caratheodory Correspondences and Minimal USCOs

Let $\mathcal{U}_{\Omega \times \mathcal{L}_Y-Y}$ denote the collection of all upper Caratheodory mappings defined on $\Omega \times \mathbb{C}_Y$ taking nonempty, $\rho_Y$-closed (and hence $\rho_Y$-compact) values in $Y$. For $\mathcal{P}(\cdot, \cdot) \in \mathcal{U}_{\Omega \times \mathcal{L}_Y-Y}$, let

$$\mathcal{U}^P := \mathcal{U}_{\Omega \times \mathcal{L}_Y-Y}[\mathcal{P}(\cdot, \cdot)]$$  

denote the collection of all upper Caratheodory mappings, $\mathcal{P}(\cdot, \cdot)$, belonging to $\mathcal{P}(\cdot, \cdot)$. Thus, $\mathcal{P}(\cdot, \cdot) \in \mathcal{U}^P$ if and only if $\mathcal{P}(\cdot, \cdot) \in \mathcal{U}_{\Omega \times \mathcal{L}_Y-Y}$ and

$$Gr\mathcal{P}(\cdot, \cdot) \subset Gr\mathcal{P}(\cdot, \cdot)$$  

for all $\omega$.

We will be interested in sub-uc-correspondences, $\mathcal{P}(\cdot, \cdot) \in \mathcal{U}^P$, with the property that for each $\omega$, $p(\omega, \cdot)$ is a minimal USCO belonging to $\mathcal{P}(\cdot, \cdot)$. Already we know that (i) for each $\omega$, $[\mathcal{P}(\omega, \cdot)]$ is nonempty (e.g., see Drewnowski and Labuda, 1990), and (ii) for each $\omega$ and $p(\omega, \cdot) \in [\mathcal{P}(\omega, \cdot)]$, $S_{p(\omega, \cdot)}$ is a $G_\delta$ set $\rho_{\mathcal{L}_Y}$-dense in $\mathcal{L}_Y \times Y$ and, therefore, for each $(\omega, v) \in \Omega \times S_{p(\omega, \cdot)}$, $p(\omega, v)$ is single-valued. What we don’t know is whether or not the uc correspondence, $\mathcal{P}(\cdot, \cdot)$, contains a sub-uc-correspondence, $p(\cdot, \cdot)$, such that for each $\omega$, $p(\omega, \cdot)$ is a minimal USCO belonging to $\mathcal{P}(\cdot, \cdot)$. We call any such sub-uc correspondence a minimal uc correspondence, and we denote by,

$$\mathcal{MUC}^P := \{ p(\cdot, \cdot) \in \mathcal{U}^P : p(\omega, \cdot) \in [\mathcal{P}(\omega, \cdot)] \} \text{ for all } \omega,$$

the collection of minimal uc correspondences belonging to $\mathcal{P}(\cdot, \cdot) \in \mathcal{U}_{\Omega \times \mathcal{L}_Y-Y}$.

Our first result shows that for any $\mathcal{P} \in \mathcal{U}_{\Omega \times \mathcal{L}_Y-Y}$,

$$\mathcal{MUC}^P$$  

is nonempty.

**Theorem 1 (Nonemptiness of MUC^P)**  
Suppose assumptions $\text{[FPP]}$ hold. For any uc correspondence, $\mathcal{P}(\cdot, \cdot) \in \mathcal{U}_{\Omega \times \mathcal{L}_Y-Y}$, there exists a uc correspondence, $p(\cdot, \cdot) \in \mathcal{U}_{\Omega \times \mathcal{L}_Y-Y}$, such that $p(\omega, \cdot) \in [\mathcal{P}(\omega, \cdot)] \text{ for all } \omega$.  

**References**

3.3 $\varepsilon$-Caratheodory Selections for Minimal UC Correspondences

We begin with the basic definitions.

**Definition 1 (Caratheodory Functions)**
Given $\mathcal{P}(\cdot, \cdot) \in \mathcal{UC}_{\infty \times \mathcal{L}_Y}$, let $p(\cdot, \cdot) \in \mathcal{MUC}_\infty^P$ (i.e., for each $\omega$, $p(\omega, \cdot) \in \{P(\omega, \cdot)\}$). A function,

$$g(\cdot, \cdot) : \Omega \times \mathcal{L}_Y^\infty \rightarrow Y,$$

is said to be Caratheodory if (i) for each $\omega \in \Omega$ the function $g(\omega, \cdot) := g_\omega(\cdot)$, defined on $\mathcal{L}_Y^\infty$ with values in $Y$ is $\rho_{\mathcal{L}_Y^\infty}$-$\rho_Y$-continuous, and if (ii) for each $v \in \mathcal{L}_Y^\infty$ the function, $g(\cdot, v) := g_v(\cdot)$ is $(B_1, B_Y)$-measurable.

**Definition 2 (Caratheodory Selections of Minimal UC Correspondences)**
Given $\mathcal{P}(\cdot, \cdot) \in \mathcal{UC}_{\infty \times \mathcal{L}_Y}$, let $p(\cdot, \cdot) \in \mathcal{MUC}_\infty^P$ (i.e., for each $\omega$, $p(\omega, \cdot) \in \{P(\omega, \cdot)\}$). A function,

$$g^\varepsilon(\cdot, \cdot) : \Omega \times \mathcal{L}_Y^\infty \rightarrow Y,$$

is said to be $\varepsilon$-Caratheodory Selection of $p(\cdot, \cdot)$ if for each $(\omega, v) \in \Omega \times \mathcal{L}_Y^\infty$ and each $(v, g^\varepsilon(\omega, v)) \in \mathcal{L}_Y^\infty \times Y$ there exists $(\overline{\omega}, \overline{v}) \in \text{Grp}(\omega, \cdot)$ such that

$$\rho_{\mathcal{L}_Y^\infty}(v, \overline{v}) + \rho_Y(g^\varepsilon(\omega, v), \overline{v}) < \varepsilon,$$

or equivalently, such that the Caratheodory function, $g^\varepsilon : \Omega \times \mathcal{L}_Y^\infty \rightarrow Y$, has the property that for each $\omega$

$$\text{Gr}(g^\varepsilon(\omega, \cdot)) \subset B_{\rho_{\mathcal{L}_Y^\infty \times Y}}(\varepsilon, \text{Grp}(\omega, \cdot)).$$

We say that the minimal uc correspondence, $p(\cdot, \cdot) \in \mathcal{MUC}_\infty^P$, is Caratheodory approximable if for each $\varepsilon > 0$, $p(\cdot, \cdot)$ has an $\varepsilon$-Caratheodory Selection.

By Theorem 4.2 in Kucia and Nowak (2000), a sufficient condition for $p(\cdot, \cdot) \in \mathcal{MUC}_\infty^P$ to have for each $\varepsilon > 0$ an $\varepsilon$-Caratheodory selection is for each $\omega$ the minimal USCO, $p(\omega, \cdot)$, to have an $\varepsilon$-continuous selection.

**Definition 3 (Continuous Selections of Minimal USCOs, $p(\omega, \cdot)$)**
Given $\mathcal{P}(\cdot, \cdot) \in \mathcal{UC}_{\infty \times \mathcal{L}_Y}$, let $p(\omega, \cdot) \in \mathcal{MUC}_\infty^P$ (i.e., for each $\omega$, $p(\omega, \cdot) \in \{P(\omega, \cdot)\}$). A function,

$$f^\varepsilon(\cdot) : \mathcal{L}_Y^\infty \rightarrow Y,$$

is said to be $\varepsilon$-continuous selection of $p(\omega, \cdot)$ if for each $v \in \mathcal{L}_Y^\infty$ and each $(v, f^\varepsilon(v)) \in \mathcal{L}_Y^\infty \times Y$ there exists $(\overline{v}, \overline{v}) \in \text{Grp}(\omega, \cdot)$ such that

$$\rho_{\mathcal{L}_Y^\infty}(v, \overline{v}) + \rho_Y(f^\varepsilon(v), \overline{v}) < \varepsilon,$$

or equivalently, such that the continuous function, $f^\varepsilon : \mathcal{L}_Y^\infty \rightarrow Y$, has the property that

$$\text{Gr}(f^\varepsilon(\cdot)) \subset B_{\rho_{\mathcal{L}_Y^\infty \times Y}}(\varepsilon, \text{Grp}(\omega, \cdot))$$

We say that the minimal uc correspondence, $p(\cdot, \cdot) \in \mathcal{MUC}_\infty^P$, is approximable if for each $\omega$ and for each $\varepsilon > 0$, the minimal USCO, $p(\omega, \cdot)$, has an $\varepsilon$-continuous selection.

Given assumptions [FPP], it follows from Theorem 5.12 in Gorneiewicz, Granas, and Kryszewski (1991) that if $p(\omega, \cdot)$ is $R_3$-valued, then for any $\varepsilon > 0$, $p(\omega, \cdot)$ has an $\varepsilon$-continuous selection. One of the main objectives of the paper is to identify conditions the correspondence $p(\omega, \cdot)$ must satisfy in order to guarantee that $p(\omega, \cdot)$ is $R_3$-valued. Under the assumptions [FPP], we will show that if $p(\cdot, \cdot) \in \mathcal{MUC}_\infty^P$ is $3$M at $(\omega, v)$ for all $(\omega, v) \in \Omega \times \mathcal{L}_Y^\infty$, then $p(\cdot, \cdot)$ takes minimally essential continuum values (and conversely). Moreover, because these continuum values are contained in $Y$, a globally arc-smooth continuum (i.e., a continuum that is arc-smooth at each of its points), $p(\cdot, \cdot)$ takes $R_3$-values.

4 Essential Sets, the $3M$ Property, and Continuum Values

We begin with the definitions of essential and minimally essential values belonging to an upper Caratheodory correspondence’s USCO part for a particular $\omega$ - where essential and minimally essential are in the sense of Fort (1950), Kinoshita (1952), and Jiang (1962). We also reintroduce the $3M$ property of Page and Resende (2011). We then show that, under assumptions [FPP], all $3M$ minimal uc correspondences are minimally essentially valued and if they have the $3M$ property, they are continuum valued, and conversely, if they are continuum valued, they are $3M$.

4.1 Essential Sets and the $3M$ Property
Let $\mathcal{P}(\cdot, \cdot)$ be an upper Caratheodory correspondence with USCO part,

$$\mathcal{P}^{USCO} = \{p(\omega, \cdot) \in \mathcal{U}_{\mathcal{L}_Y} : \omega \in \Omega\}.$$

**Definition 4 (Essential Sets)**

1. (Essential Set) A nonempty closed subset $e(\omega, v)$ of $\mathcal{P}(\omega, v)$ is said to be essential for $p(\omega, \cdot)$ at $v$ if for each $\varepsilon > 0$ there exists $\delta^\varepsilon > 0$ such that for all $v \in B_{\rho_{\mathcal{L}_Y^\infty}}(\delta^\varepsilon, v)$,

$$\mathcal{P}(\omega, v) \cap B_{\rho_Y}(e(\omega, v), 0) \neq \emptyset.$$

We will denote by $\mathcal{E}[\mathcal{P}(\omega, v)] \subset 2^{\mathcal{P}(\omega, v)}$ the collection of all nonempty, $\rho_Y$-closed subsets of $\mathcal{P}(\omega, v)$ essential for $p(\omega, \cdot)$ at $v$.

2. (Minimal Essential Set) A nonempty closed subset $m(\omega, v)$ of $\mathcal{P}(\omega, v)$ is said to be minimally essential for $p(\omega, \cdot)$ at $v$ if (i) $m(\omega, v) \in \mathcal{E}[\mathcal{P}(\omega, v)]$ and if...
(ii) $m_\omega(v^0)$ is a minimal element of $\mathcal{E}[\mathcal{P}(\omega, v^0)]$ ordered by set inclusion (i.e., if $e_\omega(v^0) \in \mathcal{E}[\mathcal{P}(\omega, v^0)]$ and $e_\omega(v^0) \subseteq m_\omega(v^0)$ then $e_\omega(v^0) = m_\omega(v^0)$). We will denote by $\mathcal{E}^*[\mathcal{P}(\omega, v^0)]$ the collection of all nonempty, closed subsets of $\mathcal{P}(\omega, v^0)$ minimally essential for $\mathcal{P}_\omega$ at $v^0 \in \mathcal{C}_\mathcal{Y}^\omega$.

The $3M$ property (i.e., the 3 misses property) is defined as follows:

**Definition 5 (The $3M$ Property)**

Let $\mathcal{P}(\cdot) \in \mathcal{UC}_{\Omega \times \mathcal{L}_C^\omega \times \mathcal{Y}}$ be a uc correspondence and consider the minimal uc correspondence, $p(\cdot) \in \mathcal{MUC}_C^\omega$. We say that $p(\cdot)$ is $3M$ at $(\omega^0, v^0)$ if the minimal $\omega^0$-USCO, $p_\omega$, has the property at $v^0$ that for any $\delta > 0$ and for any pair of nonempty disjoint closed sets, $F^1$ and $F^2$, in $Y$ there exists $v^1$ and $v^2$ in $B_{p_\omega}(\delta, v^0)$ such that

$$p_\omega(v^1) \cap F^1 = \emptyset \text{ and } p_\omega(v^2) \cap F^2 = \emptyset,$$

(i.e., $p_\omega(\cdot)$ misses $F^1$ at $v^1$ and misses $F^2$ at $v^2$ for some $v^1$ and $v^2$ in $B_{p_\omega}(\delta, v^0)$), then there exists a third point, $v^3$, in the larger open ball, $B_{p_\omega}(3\delta, v^0)$, such that

$$p_\omega(v^3) \cap [F^1 \cup F^2] = \emptyset,$$

(i.e., $p_\omega(\cdot)$ misses $F^1 \cup F^2$ at $v^3$ in $B_{p_\omega}(3\delta, v^0)$).

We say that a uc correspondence, $\mathcal{P}(\cdot)$, is $3M$ if for some $p(\cdot) \in \mathcal{MUC}_C^\omega$, $p(\cdot)$ is $3M$ at $(\omega, v)$ for all $(\omega, v) \in \Omega \times \mathcal{L}_C^\omega$. We will denote by $\mathcal{MUC}_{3M}^\omega$ the collection of all $3M$ minimal uc correspondences belonging to $\mathcal{P}(\cdot)$, and we will denote by $\mathcal{UC}_{3M}^\omega$ the collection of all uc correspondences such that $\mathcal{MUC}_{3M}^\omega \neq \emptyset$.

We say that a uc correspondence, $\mathcal{P}(\cdot)$, is quasi-minimal at $\omega$, if the $\omega$-USCO, $\mathcal{P}_\omega(\cdot) := \mathcal{P}(\omega, \cdot)$, is quasi-minimal. We say that $\mathcal{P}(\cdot)$ is quasi-minimal if the $\omega$-USCO, $\mathcal{P}_\omega(\cdot) := \mathcal{P}(\omega, \cdot)$, is quasi-minimal for all $\omega$. We will denote the collection of all quasi-minimal upper Caratheodory correspondences by $\mathcal{QUC}_{\Omega \times \mathcal{L}_C^\omega \times \mathcal{Y}}$. The following theorem establishes a fundamental fact about quasi-minimal uc correspondences: any minimal uc correspondence belonging to a quasi-minimal uc correspondence takes minimally essential values.

**Theorem 2 (The Quasi-Minimal Theorem for UC Correspondences)**

Suppose assumptions [FPP] hold. Let $\mathcal{P}(\cdot)$ be a quasi-minimal uc correspondence. Then, for $p(\cdot) \in \mathcal{MUC}_C^\omega$, $p(\omega, v) \in \mathcal{E}^*[\mathcal{P}(\omega, v)]$ for each $(\omega, v) \in \Omega \times \mathcal{L}_C^\omega$.

If $\mathcal{P}(\cdot) \in \mathcal{UC}_{\Omega \times \mathcal{L}_C^\omega \times \mathcal{Y}}$ is quasi-minimal, then it can contain only one sub uc correspondence, namely, its minimal uc correspondence.

**Theorem 3 (Quasi-Minimal USCOs and Minimal USCOs)**

Suppose assumptions [FPP] hold. Let $\mathcal{P}(\cdot)$ be a quasi-minimal uc correspondence. If $p(\cdot) \in \mathcal{MUC}_C^\omega$, then for any $\phi \in \mathcal{U}_{\mathcal{C}_\mathcal{Y}}$, with $Gr\phi$ is a proper subset of $Gr\mathcal{P}_\omega$ for some $\omega$, $Gr\phi = Gr\mathcal{P}_\omega$.

Let $\mathcal{MUC}_{3M}^\omega$ denote the collection of minimal uc correspondences, $p(\cdot)$, belonging to $\mathcal{P} \in \mathcal{UC}_{\Omega \times \mathcal{L}_C^\omega \times \mathcal{Y}}$ such that for each $\omega \in \Omega$, $p(\omega, \cdot)$, an USCO defined on $\mathcal{L}_C^\omega$ taking nonempty closed values in $Y$, is $3M$. Let $\mathcal{MUC}_{3M}^\omega(\mathcal{Y})$ denote the collection of minimal uc correspondences, $p(\cdot)$, belonging to $\mathcal{P} \in \mathcal{UC}_{\Omega \times \mathcal{C}_\mathcal{Y} \times \mathcal{Y}}$ such that for each $\omega \in \Omega$, $p(\omega, \cdot)$, an USCO defined on $\mathcal{L}_C^\omega$ taking nonempty closed and connected values in $Y$ - i.e., values in $C(Y)$, where $C(Y)$ denotes the hyperspace of nonempty closed and connected subsets of $Y$ (i.e., subsouub set of $Y$).

**Theorem 4 (For $\mathcal{P}(\cdot) \in \mathcal{UC}_{\Omega \times \mathcal{L}_C^\omega \times \mathcal{Y}}$, $\mathcal{MUC}_{3M}^\omega = \mathcal{MUC}_{3M}^\omega(\mathcal{Y})$)**

Let $(\Omega, B_\Omega)$ be a measurable space of states with Borel $\sigma$-field $B_\Omega$ and let $(\mathcal{L}_C^\omega, \rho_{\mathcal{L}_C^\omega})$ and $(\mathcal{Y}, \rho_{\mathcal{Y}})$ be Peano continua with $\mathcal{L}_C^\omega$ metrics $\rho_{\mathcal{L}_C^\omega}$ and $\rho_{\mathcal{Y}}$. Given any $\mathcal{P}(\cdot) \in \mathcal{UC}_{\Omega \times \mathcal{L}_C^\omega \times \mathcal{Y}}$, any $p(\cdot) \in \mathcal{MUC}_C^\omega$, and any state-parameter pair, $(\omega, v) \in \Omega \times \mathcal{L}_C^\omega$, the following statements are equivalent:

\[
\begin{align*}
(1) & \ p_\omega(\cdot) \text{ is } 3M \text{ at } v. \\
(2) & \ p_\omega(v) \in \mathcal{E}^*[p_\omega(v)] \cap C(Y).
\end{align*}
\]

4.2 3M Minimal Upper Caratheodory Correspondences Are $R_\delta$-Valued

First, recall the formal definition of an $R_\delta$ set.

**Definition 6 (R_\delta-Sets)**

A set $M$ is called an $R_\delta$-set, denoted $M \in R_\delta$, if there exists a sequence of compact, nonempty contractible sets, $\{M^n\}_n$, such that

$$M^{n+1} \subseteq M^n \text{ for every } n$$

and

$$M = \bigcap_{n=1}^\infty M^n.$$

A key fact allowing us to show that 3M minimal uc correspondences, $p(\cdot) \in \mathcal{MUC}_C^\omega$, are $R_\delta$-valued is that each such minimal uc correspondences is continuum-valued and has minimally essential values contained in a globally arc-smooth continuum, $Y \subseteq \mathbb{R}^m$ (see Section...
25.8 in Illanes and Nadler, 1999). By Lemma 19.1 in Illanes and Nadler (1999), each subcontinuum, \( p(\omega, v) \), contained in \( Y \) is given by the intersection of a decreasing sequence of Peano continua, \( \{B^n\}_n \), contained in \( Y \), and therefore, the intersection of a decreasing sequence of \( M \)-convex continua contained in \( Y \). Thus, if we can show that each Peano continua, \( B^n \subset Y \), making up the decreasing sequence of Peano continua, \( \{B^n\}_n \), such that \( p(\omega, v) = \cap_n B^n \), is contractible, then by the definition of \( R_3 \)-valuedness, \( p(\omega, v) \) is an \( R_3 \)-set. We have the following result:

**Theorem 5 (The \( R_3 \) Theorem: for each \( (\omega, v) \), \( p(\omega, v) \) is an \( R_3 \)-set)**

Suppose assumptions [FPP] hold. Let \( \mathcal{P}(\cdot, \cdot) \) be a uc correspondence. If \( p(\cdot, \cdot) \in \text{MUC}^\mathcal{P}_{3M} \), then for each \( (\omega, v) \in \Omega \times \mathcal{L}^\mathcal{X} \), \( p(\omega, v) \) is an \( R_3 \)-set.

5 Caratheodory Approximable 3M Minimal UC Correspondences

Our main result on 3M minimal uc correspondences and Caratheodory approximation is the following:

**Theorem 6 (Sufficient Conditions for Caratheodory Approximability)**

Suppose assumptions [FPP] hold. Let \( \mathcal{P}(\cdot, \cdot) \) be a 3M uc correspondence. If \( p(\cdot, \cdot) \in \text{MUC}^\mathcal{P}_{3M} \), then \( p(\cdot, \cdot) \) is Caratheodory approximable.

**Proof:** In light of Theorem 4.2 in Kucia and Nowak, (2000), it suffices to show that for each \( \omega \in \Omega \), the minimal USCO \( p(\omega, \cdot) \) is approximable. By Theorem above, we have that for each \( \omega \), the minimal USCO, \( p(\omega, \cdot) \in [\mathcal{P}_\omega(\cdot)] \) is \( R_3 \)-valued. By Corollary 5.6 in Gorniewicz, Granas, and Kryszewski (1991), because \( p(\omega, \cdot) \) is a mapping defined on the ANR (absolute neighborhood retract) space \( \mathcal{L}_Y^\mathcal{X} \) taking nonempty, compact, \( R_3 \) valued in the ANR space \( Y \), the minimal USCO, \( p(\omega, \cdot) \), is \( \infty \)-proximally connected valued, and therefore a \( J \) mapping. Thus, by Theorem 5.12 in Gorniewicz, Granas, and Kryszewski (1991), \( p(\omega, \cdot) \) is approximable. Because this is true for each \( \omega \in \Omega \), by Theorem 4.2 in Kucia and Nowak (2000), \( p(\cdot, \cdot) \) is Caratheodory approximable.

Q.E.D.

If, for example, \( \mathcal{P}(\cdot, \cdot) \) is such that for each \( \omega \), \( \mathcal{P}(\omega, \cdot) \) contains a convex-valued or star-shape valued minimal USCO, \( p(\omega, \cdot) \), and thus, an \( R_3 \)-valued minimal USCO - then \( \mathcal{P}(\cdot, \cdot) \) is approximable. In addition, if for each \( \omega \), \( \mathcal{P}(\omega, \cdot) \) contains a minimal USCO, \( p(\omega, \cdot) \), taking arc-like continuum values, arc-smooth continuum values, or dendritic values, then \( p(\omega, \cdot) \) is contractively-valued - and thus, \( R_3 \)-valued (for related results see Cellina, 1969 and Beer, 1983, 1988 and 1993) - implying, via Kucia and Nowak (2000) that \( \mathcal{P}(\cdot, \cdot) \) is Caratheodory approximable.

6 Selections and Fixed Points of 3M UC Correspondences

Many, if not all, parameterized, state-contingent \( m \)-person noncooperative games have globally arc-smooth payoff spaces. Our main selection result for 3M uc correspondences mapping into globally arc-smooth continua is the following:

**Theorem 7 (A Selection Result for 3M UC Correspondences Mapping into Globally Arc-Smooth Continua)**

Suppose assumptions [FPP] hold. Let \( \mathcal{P}(\cdot, \cdot) : \Omega \times \mathcal{L}^\mathcal{X} \rightarrow 2^Y \) be a uc correspondence taking nonempty closed values in the everywhere arc-smooth continuum \( Y := [-M, M]^m \subset R^m \).

If \( \text{MUC}^\mathcal{P}_{3M} \neq \emptyset \), then there exists \( v^* \in \mathcal{L}_Y^\mathcal{X} \) such that \( v^*(\omega) \in \mathcal{P}(\omega, v^*) \) a.e. \( [\mu] \).

An immediate Corollary of Theorem 7 is the following fixed point result.

**Theorem 8 (Fixed Points for Measurable-Selection-Valued Correspondences Induced by 3M Caratheodory Compositions)**

Let \( (\omega, v) \rightarrow \mathcal{P}(\omega, v) \) be a Caratheodory composition. If \( \mathcal{P}(\cdot, \cdot) \) is 3M, then \( v \rightarrow \mathcal{S}^\infty(\mathcal{P}_v) \) has fixed points (i.e., there exists \( v^* \in \mathcal{L}_Y^\mathcal{X} \) such that \( v^* \in \mathcal{S}^\infty(\mathcal{P}_v) \)).

References