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$(d,3)$-track layouts of bipartite graph subdivisions

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Abstract:

A (d,k)-*track layout* of a graph G consists of a k-*track assignment* of G and an edge d-coloring of G with no monochromatic X-*crossing*. This paper studies the problem of $(d,3)$-track layout of bipartite graph subdivisions. As for track layout, V. Dujmović and D. R. Wood showed that every graph G with n vertices has a $(d,3)$-track subdivision of G with $2\lceil \log_d \mathrm{qn}(G)\rceil + 1$ division vertices per edge, where $\mathrm{qn}(G)$ is the queue number of G. This paper improves their result for the case of bipartite graphs, and shows that for every integer $d \geq 2$, every bipartite graph $G_{m,n}$ has a $(d,3)$-track subdivision of $G_{m,n}$ with $\lceil \log_d n\rceil - 1$ division vertices per edge, where m and n are numbers of vertices of the partite sets of $G_{m,n}$ with $m \geq n$.

Keywords: graph layout, bipartite graph, subdivision, track layout

1 Introduction

A graph $G_{m,n}$ is a *bipartite graph* having *partite sets* A with m vertices and B with n vertices if $V(G) = A \cup B$, $A \cap B = \emptyset$ and each edge joins a vertex of A with a vertex of B. A bipartite graph $G_{m,n}$ is *complete* if $G_{m,n}$ contains all edges joining vertices in distinct sets. A complete bipartite graph is denoted by $K_{m,n}$.

An *ordering* of a set S is a total order $<_\sigma$ on S. It will be convenient to interchange "σ" and $<_\sigma$ when there is no ambiguity. A *vertex ordering* of a graph G is an ordering σ of the vertex set $V(G)$.

A *vertex k-coloring* of a graph G is a partition $\{V_i : 1 \leq i \leq k\}$ of $V(G)$ such that for every edge $(v,w) \in E(G)$, if $v \in V_i$ and $w \in V_j$ then $i \neq j$. Suppose that each color class V_i is ordered by $<_i$. Then the ordered set $(V_i, <_i)$ is called a *track*, and $\{(V_i, <_i) : 1 \leq i \leq k\}$ is a *k-track assignment* of G.

The *span* of an edge (v,w) in a track assignment $\{(V_i, <_i) : 1 \leq i \leq k\}$ is $\mid i - j \mid$ where $v \in V_i$ and $w \in V_j$. An *X-crossing* in a track assignment consists of two edges (v,w) and (x,y) such that $v, x \in V_i$, $w, y \in V_j$, $v <_i x$ and $y <_j w$, for distinct colors i and j. An *edge d-coloring* of G is simply a partition $\{E_i : 1 \leq i \leq d\}$ of $E(G)$. An edge $(v,w) \in E_i$ is said to be *colored* i. A *(d,k)-track layout* of G consists of a k-track assignment of G and an edge d-coloring of G with no monochromatic X-crossing. A graph admitting a (d,k)-track layout is called a *(d,k)-track graph*. The minimum k such that a graph G is a (d,k)-track graph is denoted by $\mathrm{tn}_d(G)$. A $(1,k)$-track layout is called a k-track layout. A graph admitting a k-track layout is called a k-track graph. The *track-number* of G is $\mathrm{tn}_1(G)$, simply denoted by $\mathrm{tn}(G)$. For a summary of bounds on the track-number see papers [1, 2].

A *subdivision* of a graph G is a graph obtained from G by replacing each edge $(v,w) \in E(G)$ by a path with at least one edge whose endpoints are v and w. Internal vertices on this path are called *division vertices*, while v and w are called *original vertices*.

This paper studies track layouts of graph subdivisions. V. Dujmović and D. R. Wood [3] showed the following theorem:

Theorem 1 (V. Dujmović and D. R. Wood [**3**]) *For every integer $d \geq 2$, every graph G has a $(d,3)$-track subdivision of G with $2\lceil \log_d \mathrm{qn}(G)\rceil + 1$ division vertices per edge, where $\mathrm{qn}(G)$ is the queue number of G.*

The definition of the *queue number* is as follows. In a vertex ordering σ of a graph G, let $L(e)$ and $R(e)$ denote the endpoints of each edge $e \in E(G)$ such that $L(e) <_\sigma R(e)$. Consider two edges $e, f \in E(G)$ with no common endpoint such that $L(e) <_\sigma L(f)$. If $L(e) <_\sigma L(f) <_\sigma R(f) <_\sigma R(e)$ then e and f *nest*. A *queue* is a set of edges $E \subset E(G)$ such that no two edges in E nest. For an integer $d > 0$, a *d-queue layout of G* consists of a vertex ordering σ of G and a partition $\{E_i : 1 \leq i \leq d\}$ of $E(G)$ such that each E_i is a queue in σ. The *queue number* $\mathrm{qn}(G)$ of a graph G is the minimum d such that there is a d-queue layout of G. As for queue layout see [5, 6] etc. There is a summary of bounds on the queue numbers for various kinds of graph families in [7].

As for Theorem 1, V. Dujmović and D. R. Wood [3] also showed that the order of the number of division vertices is optimal. Thus, to find a track layout with fewer division vertices for various kinds of graph families become an interesting problem.

This paper deals with the number of division vertices of bipartite graphs. For the queue number of a complete bipartite graph $K_{m,n}$, L. S. Heath and A. L. Rosenberg **[6]** showed the following theorem:

Theorem 2 (L. S. Heath and A. L. Rosenberg **[6]**)

$$\mathrm{qn}(K_{m,n}) = \min\left(\lceil m/2 \rceil, \lceil n/2 \rceil\right).$$

Applying Theorem 2 to Theorem 1, we have the following corollary:

Corollary 3 *For every integer $d \geq 2$, every complete bipartite graph $K_{m,n}$ ($m \geq n$) has a $(d,3)$-track subdivision with*

$$2\lceil \log_d \lceil n/2 \rceil \rceil + 1 = \begin{cases} 2\lceil \log_d (n+1) - \log_d 2 \rceil + 1 & \textit{(if } n \textit{ is odd)} \\ 2\lceil \log_d n - \log_d 2 \rceil + 1 & \textit{(if } n \textit{ is even)} \end{cases}$$

division vertices per edge.

This paper improves this result by showing the following theorem:

Theorem 4 *For every integer $d \geq 2$, every bipartite graph $G_{m,n}$ has a $(d,3)$-track subdivision with $\lceil \log_d n \rceil - 1$ division vertices per edge, where $m \geq n$.*

Here, we consider a graph layout named prism layouts for graphs. A *triangular prism layout for graphs* is a graph layout on a triangular prism that carries the vertices along the three crests between two triangles of the prism and the edges in the three rectangular surfaces such that no two edges with same color cross in the interior of the surfaces. Also, a *topological* prism layout for graphs is defined so that edges are allowed to cross the crests. As for topological prism layouts, it is desirable to have good bounds on number of edge-crossings over crests for various classes of graphs. Then a $(d,3)$-track layout of graph subdivisions can be regarded as a d-color-edge topological triangular prism layout for graphs. And this paper constructs topological triangular prism layouts for complete bipartite graphs with fewer edge-crossings over crests than previous results.

The proof of Theorem 4 is similar to that of Theorem 1 (V. Dujmović and D. R. Wood [3]), however, it becomes simpler by capitalizing the character of bipartite graphs.

In section 2, we define a breadth-first ordering recursively on a string set. In section 3, we consists $(d,3)$-track layouts for complete bipartite graph subdivisions with fewer division vertices than previous results.

2 A breadth-first ordering on a strings-set

In this section, we define a breadth-first ordering recursively on a strings set. This definition plays an important role when we prove the main theorem.

Let $S = \{0, \cdots, d-1\}$ be the d-adic alphabet and S^* the set of all strings over S of length at most h $(h > 0)$. If $s \in S^*$ has length k $(0 \le k \le h)$, then write

$$s = s_1 s_2 \ldots s_k$$

where s_i is the character of s in position i. Order the elements of S by $0 < \cdots < d-1$. Define a *breadth-first ordering* $<_*$ on S^* as follows: For $s = s_1 \ldots s_i, t = t_1 \ldots t_j \in S^*$, define $s <_* t$ when either of the following conditions holds:

1. $i < j$.
2. $i = j$, $s_1 \ldots s_{i-1} = t_1 \ldots t_{i-1}$, $s_i < t_i$.
3. $i = j (> 1)$, $s_1 \ldots s_{i-1} <_* t_1 \ldots t_{i-1}$.

As an example, if $d = 3$, the strings of length at most 2 are ordered as follows:

$$\epsilon <_* 0 <_* 1 <_* 2 <_* 00 <_* 01 <_* 02$$

$$<_* 10 <_* 11 <_* 12 <_* 20 <_* 21 <_* 22$$

where ϵ denotes the empty string.

3 Proof of Theorem 4

In this section, we will construct a $(d, 3)$-track layout of a subdivision of $G_{m,n}$ with $\lceil \log_d n \rceil - 1$ division vertices per edge where $d \ge 2$.

Define $k = \lceil \log_d n \rceil$. A number s $(0 \le s < n)$ has a unique representation as a string in $S^k \subseteq S^*$ using d-adic representation, where S^k is the set of all elements of length k. For a number s, use the representation $s_1 \ldots s_k$ for its d-adic representation, where s_1 is the highest-order digit. For a string $s = s_1 \ldots s_k$ in S^k let $s(i)$ be the string consisting of the first i letters of s, that is,

$$s(i) = s_1 \ldots s_i$$

and $s(0)$ be the empty string ϵ.

Consider a subdivision $G^*_{m,n}$ of $G_{m,n}$ made by subdividing each edge

$$(a_s, b_t) \in E(G_{m,n})$$

$$(a_s \in A, b_t \in B, 0 \le s < m, 0 \le t < n)$$

by adding $k - 1$ vertices between a_s and b_t. We label these vertices in the order from a_s to b_t as follows;

$$a_s = (a_s, b_t; 0), (a_s, b_t; 1), \ldots, (a_s, b_t; k) = b_t.$$

where $(a_s, b_t; 0)$ is identified with a_s and $(a_s, b_t; k)$ is identified with b_t.

Since a $(d, 3)$-track layout of $G^*_{m,n}$ corresponds to a $(d, 3)$-track subdivision of $G_{m,n}$ by regarding vertices $V(G^*_{m,n}) - V(G_{m,n})$ as division vertices, we will construct a $(d, 3)$-track layout of $G^*_{m,n}$. First we define a vertex ordering σ of $V(G^*_{m,n})$ and then add d numbers, 0, 1, …, $d-1$ to edges of $G^*_{m,n}$ so that there is no monochromatic X-crossing.

For two vertices $(a_s, b_t; i), (a_p, b_q; j) \in V(G^*_{m,n})$, we define $(a_s, b_t; i) <_\sigma (a_p, b_q; j)$ when one of the following three conditions holds:

1. $t(i) <_* q(j)$.
2. $t(i) = q(j)$ and $s < p$.

3. $t(i) = q(j)$ and $s = p$, $t < q$.

Lemma 5 *The ordering σ is a total order.*

Proof. Obviously, σ is a partial order. Thus to show that σ is a total order, we show the following statement: For the vertex ordering σ and any two vertices $(a_s, b_t; i), (a_p, b_q; j) \in V(G^*_{m,n})$, either

$$(a_s, b_t; i) \leq_\sigma (a_p, b_q; j) \text{ or } (a_p, b_q; j) \leq_\sigma (a_s, b_t; i) \quad (*)$$

holds.

If $i \neq j$, then by the definition of the breadth-first ordering either

$$t(i) <_* q(j) \text{ or } q(j) <_* t(i)$$

holds. Hence from the definition 1 of σ, we have the statement $(*)$. If $i = j$, then for two strings $t(i)$ and $q(j)$ either

$$t(i) <_* q(j), q(j) <_* t(i) \text{ or } t(i) = q(j)$$

holds. If $t(i) <_* q(j)$ or $q(j) <_* t(i)$, then from the definition 1 of σ, we have the statement $(*)$. If $t(i) = q(j)$, then for two numbers s and p either

$$s < p, s > p \text{ or } s = p$$

holds. If $s < p$ or $s > p$, then from the definition 2 of σ, we have the statement $(*)$. Suppose $s = p$. For two numbers t and q either

$$t < q, t > q \text{ or } t = q$$

holds. If $t < q$ or $t > q$, then from the definition 3 of σ, we have the statement $(*)$. If $t = q$, then $(a_s, b_t; i) = (a_p, b_q; j)$. Therefore, the ordering σ is a total order. ■

As for an edge d-coloring of $G^*_{m,n}$, let an edge $((a_s, b_t; i-1), (a_s, b_t; i))$ be colored t_i.

Lemma 6 *Let $V_i = \{(a_s, b_t; i) : a_s \in A, b_t \in B, 0 \leq s < m, 0 \leq t < n\}$ for $0 \leq i \leq k$. For every bipartite graph $G_{m,n}$, the subdivision $G^*_{m,n}$ of $G_{m,n}$ has the $(d, k+1)$-track layout defined by the family of the ordered sets $\{(V_i, \sigma |_{V_i}) : 0 \leq i \leq k\}$ and the edge d-coloring we mentioned above. Moreover the maximum span is one, and the number of division veritices per edge is $\lceil \log_d n \rceil - 1$.*

Proof. By Lemma 5, the ordering σ is a total order. Thus the family of the ordered sets $\{(V_i, \sigma) : 0 \leq i \leq k\}$ is a $k+1$-track assignment of $G^*_{m,n}$.

Next, we show that this track layout is legal, i.e., no two edges in this track assignment $\{(V_i, \sigma)\}$ form a monochromatic X-crossing.

Let $((a_s, b_t; i-1), (a_s, b_t; i))$ and $((a_p, b_q; j-1), (a_p, b_q; j))$ be two edges in $E(G^*_{m,n})$ $(0 < i, j \leq k)$ that form an X-crossing. Note that by the definition of the vertex ordering,

$$(a_s, b_t; i-1) <_\sigma (a_s, b_t; i) \text{ and } (a_p, b_q; j-1) <_\sigma (a_p, b_q; j).$$

We may assume that the endpoints of the two edges are laid out from left to right in the order

$$(a_s, b_t; i-1) <_\sigma (a_p, b_q; j-1) \quad \text{and} \quad (a_p, b_q; j) <_\sigma (a_s, b_t; i). \qquad (**)$$

We want to show that the two division edges have different colors, that is, $t_i \neq q_j$. From the assumption $(**)$ and the definition of the vertex-ordering, we have

$$t(i-1) \leq_* q(j-1) \text{ and } q(j) \leq_* t(i).$$

These inequalities hold only when $i = j$. Suppose $t(i-1) <_* q(i-1)$. Then by the definition of the vertex ordering we have $t(i) <_* q(i)$ which contradicts the assumption. Thus we have $t(i-1) = q(i-1)$ and $q(i) \leq_* t(i)$. If $q_i = t_i$, then $q(i) = t(i)$. By the definition 2 or 3 of the vertex ordering σ and

$$(a_s, b_t; i-1) <_\sigma (a_p, b_q; i-1),$$

we have either $s < p$ or $s = p, t < q$, respectively. Thus, we have

$$(a_s, b_t; i) <_\sigma (a_p, b_q; i),$$

which contradicts the assumption $(**)$. Therefore $q_i \neq t_i$. Therefore we have proved that this track layout is legal.

Moreover by the definition of the adjacency relations we can easily find that the maximum span of the $(d, k+1)$-track layout of $G^*_{m,n}$ is one. Also, each edge (a_s, b_t) of $G_{m,n}$ is divided by adding $\lceil \log_d n \rceil - 1$ division points in the subdivision $G^*_{m,n}$. Thus, we have Lemma 6. ■

The following “wrapping” algorithm (Lemma 7) is implicitly proved by Felsner, Liotta, and Wismath [4] and generalized by V. Dujmović, A. Por, and D. R. Wood **[3]** to span s $(s \geq 1)$.

Lemma 7 (V. Dujmović, A. Por, and D. R. Wood **[3]**) *If a graph G has a (d, k)-track layout with maximum span one, then G has a $(d, 3)$-track layout.*

Applying Lemma 6 to Lemma 7, we find that $G^*_{m,n}$ has a $(d, 3)$-track layout. Moreover, applying wrapping algorithm used in the proof of Lemma 7 to the $(d, k+1)$-track layout which we construct in the proof of Lemma 6, we can prove Theorem 4.

Proof of Theorem 4. Construct a vertex three-coloring of $G^*_{m,n}$ by merging tracks $\{V_i : i \equiv j (\mathrm{mod} 3)\}$ for each j, $(0 \leq j < 3)$. Then we have $(d, 3)$-track assignment $\{(V_i, \sigma) : i = 0, 1, 2\}$ of $G^*_{m,n}$. We show that this $(d, 3)$-track assignment and the edge-coloring we defined above form a $(d, 3)$-track layout.

Let $((a_s, b_t; i-1), (a_s, b_t; i))$ and $((a_p, b_q; j-1), (a_p, b_q; j))$ be two edges in $E(G^*_{m,n})$ $(0 < i, j \leq k)$ that form an X-crossing. We may assume that the endpoints of the two edges are laid out from left to right in the order

$$(a_s, b_t; i-1) <_\sigma (a_p, b_q; j-1) \text{ and } (a_p, b_q; j) <_\sigma (a_s, b_t; i).$$

From the above assumption and the definition of the vertex ordering, this inequality holds only when $i = j$. In this case, these two edges in the $(d, 3)$-track layout are laid out in the same way as in the original (d, k)-track layout. Therefore these two edges do not form a monochromatic X-crossing.

This wrapping algorithm does not change the number of division vertices for each edge, thus this $(d, 3)$-track layout also has $\lceil \log_d n \rceil - 1$ division vertices per edge. ■

4 Conclusion

This paper showed that, for every bipartite graph $G_{m,n}$ on partite sets of size m and n $(m \geq n)$, the graph obtained from $G_{m,n}$ by dividing each edge with $\lceil \log n \rceil$?1 vertices has a $(d, 3)$-track layout. This result is an improvement of a consequence of known results in the sense that the smaller number of division vertices are enough for the existence of a $(d, 3)$-track layout of a subdivision of a graph in the case where the graph is bipartite. The proof is given by constructing a concrete $(d, 3)$-track layout. We don't know whether this result is best possible or not. To find better track layouts for bipartite graphs is still an interesting problem.

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