RA-001

On (k, r)-gatherings on a Road

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Abstract Given two integers k and r, a set of customer locations C, and a set of potential facility locations F, we wish to compute an assignment A of C to F such that (1) for each $c \in C$ the distance between c and $A(c) \in F$ is at most k, and (2) for each $f \in F$ the number of customers assigned to f is either zero or at least r. Such an assignment is called *a* (k, r)*-gathering of* C *to* F. Intuitively we wish to assign customers to near "open" facilities so that each "open" facility has an enough number of customers, namely r or more customers.

In this paper we solve the problem in linear time when the customer locations and potential facility locations are on a line.

1 Introduction

Given two integers k and r, a set of customer locations C, and a set of potential facility locations F, we wish to compute an assignment A of C to F such that (1) for each $c \in C$ the distance between c and $A(c) \in F$ is at most k, and (2) for each $f \in F$ the number of customers assigned to f is either zero or at least r. Such an assignment is called *a* (k, r)*-gathering of* C *to* F. See some examples in Fig. 1 and 2. We say a facility f is *open* in an assignment if the number of customers assigned to f is at least r. Intuitively we wish to assign customers to near open facilities so that each open facility has an enough number of customers, namely r or more customers. This is a variant of the r-gathering problem in [1]. However the graph version of the problem is NP-hard even for $k = 1$ and $r = 3$ [1].

In this paper we solve the problem in linear time when the customer locations and potential facility locations are on a line.

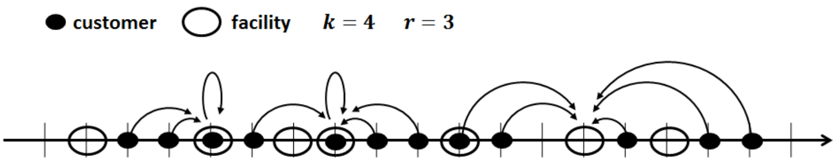

Figure 1: An example of a non-overlapping (k, r)-gathering.

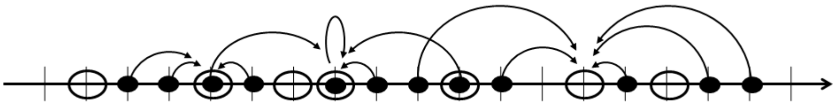

Figure 2: An example of an overlapping (k, r)-gathering.

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We regard C as the set of distinct coordinates of the customers, and F as the set of distinct coordinates of the potential facilities. In this paper we assume the elements in C and F are on the horizontal axis and sorted respectively.

If some customer has no facility within distance k then no (k, r)-gathering exists, so we assume no such customer exists in C. If some facility has less than r customers within distance k then the facility is never open in any (k, r)-gathering, so we assume no such potential facility exists in F.

2 First Algorithm

In this section we design our first algorithm for the (k, r)-gathering problem, and all $c \in C$ and $f \in F$ are integers. The algorithm is a simple dynamic programming. First we define subproblems for our dynamic programming.

Given two integers x and s, let $C(x, s)$ be the subset of C consisting of the coordinates of customers with $x + k - s$ or less, and $F(x)$ be the subset of F consisting of the coordinates of potential facilities with x or less. An assignment A of $C(x, s)$ to $F(x)$ is called *a partial gathering of* $C(x, s)$ *to* $F(x)$ if (1) A is a (k, r)-gathering of $C(x, s)$ to $F(x)$, and (2) the facility f at x is open. Intuitively this is a (k, r)-gathering of customers locating "left" of open facility f at x, in which the customers in the rightmost s locations of f are not assigned yet. A (k, r)-gathering of C to F, a solution of the original problem, exists if and only if a partial gathering of $C(x, 0)$ to $F(x)$ exists for some facility at x within distance k from the rightmost customer. Let $\mathrm{PG}(x, s)$ be the subproblem which ask for the existance of a partial gathering of $C(x, s)$ to $F(x)$.

We have the following fact.

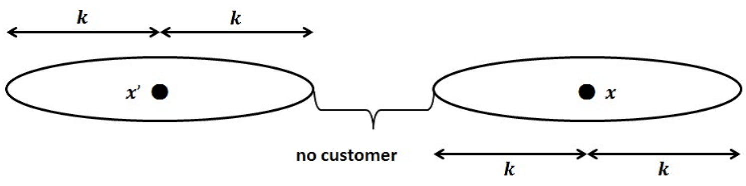
Fact 1 If $\mathrm{PG}(x, s)$ has a solution with $s > 1$, then $\mathrm{PG}(x, s-1)$ has a solution.

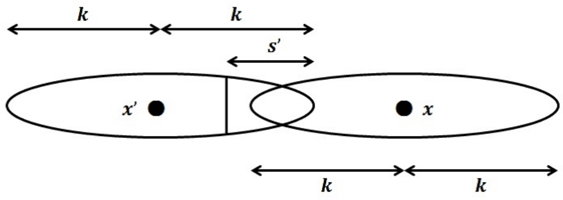
We have two lemmas. We say a solution of $\mathrm{PG}(x, s)$ is *overlapping* if for some pair of facilities the two intervals induced by the assigned customers overlap. The assignment in Fig. 1 is non-overlapping, while the assignment in Fig. 2 is overlapping.

Lemma 1 If $\mathrm{PG}(x, s)$ has a solution then $\mathrm{PG}(x, s)$ has a non-overlapping solution.

Proof We say a pair of customers (c_l, c_r) is a reverse pair in an assignment A if $A(c_l) > A(c_r)$ and $c_l < c_r$. Assume $\mathrm{PG}(x, s)$ has only overlapping solutions. Let A be a solution with the minimum number of reverse pairs. Then by swapping the assignments of a reverse pair c_l and c_r a solution with less reverse pairs is obtained. A contradiction. $\square$

Lemma 2 For each $x \in F$, $\mathrm{PG}(x, s)$ has a solution if and only if
either
(i) $|x - c| \le k$ for each $c \in C(x, s)$ and $|C(x, s)| \ge r$, or
(ii) for some $x' < x$ and $s' \in [0, 2k]$, $\mathrm{PG}(x', s')$ has a solution such that either
 (a) $x' + k < x - k$ and $(x' + k, x - k)$ has no customer location, and $[x - k, x + k - s]$ contains r or more customers,
 (b) $x' + k \ge x - k$, $x' + k - s' < x - k$ and $[x - k, x + k - s]$ contains r or more customers, or
 (c) $x' + k - s' \ge x - k$ and $(x' + k - s', x + k - s]$ contains r or more customers.





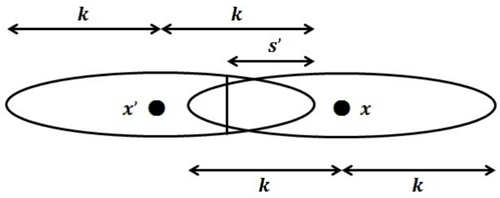


Figure 3: Illustrations for condition (ii) of Lemma 2.

Proof Assume $\mathrm{PG}(x, s)$ has a solution. Then $\mathrm{PG}(x, s)$ has a non-overlapping solution A by Lemma 1. We have two cases. If all customers are assigned to the facility at x then (i) holds. Otherwise let x' be the open facilities in A with the maximum coordinate except x. Then (ii) holds.

Assume either (i) or (ii) holds. If (i) holds then $\mathrm{PG}(x, s)$ has a simple (k, r)-gathering, in which all customers are assigned to the facility at x. If (ii) holds then by assigning the customers in either $[x - k, x + k - s]$ or $(x' + k - s', x + k - s]$ to the facility at x we can extend a solution of $\mathrm{PG}(x', s')$ to a solution of $\mathrm{PG}(x, s)$. $\square$

Intuitively above condition means $\mathrm{PG}(x, s)$ has a solution if and only if either (i) the facility at x can serve all customers and the number of customers is r or more, or (ii) some smaller subproblem $\mathrm{PG}(x', s')$ has a partial gathering A' and an assignment of customers around the facility f at x to f can be "patched" to A' to construct a partial gathering of $\mathrm{PG}(x, s)$, in which the customers in the rightmost s locations of f are reserved, and possibly to be assigned to some facility on the right.

Based on Lemma 2 we have the following algorithm. If subproblem $\mathrm{PG}(x, s)$ has a solution then the algorithm sets $\mathrm{Ans}(x, s)$ = "exists", otherwise sets $\mathrm{Ans}(x, s)$ = "not exist".

Algorithm find-gathering(C, F, k, r)

let c_{min} and c_{max} be the minimum and the maximum in C
for each facility $x \in F$ with $|x - c_{min}| \leq k$ (in increasing order)
  for each $s \in [0, 2k]$
    if $|C(x, s)| \geq r$
    then $\text{Ans}(x, s)$ = "exists"
    else $\text{Ans}(x, s)$ = "not exist"
for each facility $x \in F$ with $|x - c_{min}| > k$ (in increasing order)
  for each $s \in [0, 2k]$
    begin
      $\text{Ans}(x, s)$ = "not exist"
      for each facility x' with $x' < x$
        for each $s' \in [0, 2k]$
          if $\text{Ans}(x', s')$ = "exists" and either
            (a) $x' + k < x - k$ and $(x' + k, x - k)$ has no customer and
              $[x - k, x + k - s]$ contains r or more customers,
            (b) $x' + k \geq x - k$ and $x' + k - s' < x - k$ and
              $[x - k, x + k - s]$ contains r or more customers, or
            (c) $x' + k - s' \geq x - k$ and $(x' + k - s', x + k - s]$ contains r or more customers
          then $\text{Ans}(x, s)$ = "exists"
    end
Ans = "not exist"
for each facility $x \in F$ with $|x - c_{max}| \leq k$
  if $\text{Ans}(x, 0)$ = "exists"
  then Ans = "exists"
return Ans

As a preprocessing we prepare the following data in three arrays L, R and r. For each facility $f \in F$ we compute the total number $L(f)$ of customers less than $f - k$ and the total number $R(f)$ of customers less than or equal to $f + k$. Then $(x' + k, x - k)$ has no customer iff $R(x') = L(x)$, and we can check this in constant time. Also for each $f \in F$ we compute the location $r(f)$ of the r-th smallest customer in $[f - k, f + k]$, which always exists by the assumption in Section 1. Then $[x - k, x + k - s]$ contains r or more customers iff $r(f) \leq x + k - s$, and we can check this in constant time. One can compute L by the following algorithm as preprocessing, and also compute R and r similarly. Assume $C = \{c_0, c_1, \ldots, c_{|C|-1}\}$. However in the algorithm above we need $O(k)$ time to check if $(x' + k - s', x + k - s]$ contains r or more customers.

Algorithm $L(C, F, k)$

$j \leftarrow 0$
for each $f_i \in F$ (in increasing order)
  begin
    while $c_j < f_i - k$
      $j \leftarrow j + 1$
    $L(f_i) = j$
  end

Lemma 3 One can solve the (k, r)-gathering problem in $O(k^3|F|^2)$ time if all customers and facilities are on a line with integer coordinates.

3 Improvement I

In this section we design a faster dynamic programming algorithm. The algorithm in Section 2 computes the existance of a solution of PG(x, s) for each x and s. Let $s(x)$ be the maximum s such that PG(x, s) has a solution, and PG(x) be the subproblem which ask for $s(x)$. For convenience let $s(x) = -1$ if PG(x, s) has no solution for any $s \in [0, 2k]$. By Fact 1, PG(x, s) has a solution for each $s \leq s(x)$ and has no solution for $s > s(x)$. Thus if we have just $s(x)$ then we know whether each of PG$(x, 0)$, PG$(x, 1)$, ..., PG$(x, 2k)$ has a solution or not.

We have the following algorithm.

Algorithm find-gathering2(C, F, k, r)
let c_{min} and c_{max} be the minimum and the maximum in C
let c_r be the r-th smallest customer in C
for each facility $x_i \in F$ with $|x_i - c_{min}| \leq k$ (in increasing order)
    /* $[x_i - k, x_i + k]$ contains r or more customers */
  $s(x_i) = x_i + k - c_r$
for each facility $x_i \in F$ with $|x_i - c_{min}| > k$ (in increasing order)
  begin
    $s(x_i) = -1$
    for each facility x_j with $x_j < x_i$
      if $s(x_j) \neq -1$ and $x_j + k < x_i - k$ and $(x_j + k, x_i - k)$ has no customer
      then $s(x_i) = max\{x_i + k - c', s(x_i)\}$ where c' is the r-th smallest customer in $[x_i - k, x_i + k]$
      /* $[x_i - k, x_i + k]$ contains r or more customers */
      else if $s(x_j) \neq -1$ and $x_j + k \geq x_i - k$ and $x_j + k - s(x_j) < x_i - k$
      then $s(x_i) = max\{x_i + k - c', s(x_i)\}$ where c' is the r-th smallest customer in $[x_i - k, x_i + k]$
      else if $s(x_j) \neq -1$ and $x_j + k - s(x_j) \geq x_i - k$ and
          $(x_j + k - s(x_j), x_i + k]$ contains r or more customers
      then $s(x_i) = max\{x_i + k - c'', s(x_i)\}$
          where c'' is the r-th smallest customer in $(x_j + k - s(x_j), x_i + k]$
  end
Ans = "not exist"
for each facility $x_i \in F$ with $|x_i - c_{max}| \leq k$
  if $s(x_i) \geq 0$
  then Ans = "exists"
return Ans

In this algorithm all $c \in C$ and $x_i \in F$ are allowed to be real numbers, and not restricted to integers. We store the coordinate of $x_i \in F$ and $c_j \in C$ in an array respectively and access it with its index. For each facility $x_i \in F$ we compute the index of the smallest customer c_j with $x_i - k \leq c_j$, and the index of the largest customer c_j with $c_j \leq x_i + k$. We additionally store the index t of the customer at $x_j + k - s(x_j)$ for each $x_j \in F$. Then we can compute the r-th smallest customer in $(x_j + k - s(x_j), x_i + k]$ in constant time, since it is c_{t+r}.

Lemma 4 One can solve the (k, r)-gathering problem in $O(|F|^2)$ time if all customers and facilities are on a line.

4 Improvement II

In this section we design a linear time algorithm. Our idea is the following two lemmas. Let c_{min} be the minimum in C.

Lemma 5 For two facilities x_l and x_r, if $x_l < x_r$, $s(x_l) \neq -1$ and $s(x_r) \neq -1$, then $x_l+k-s(x_l) \leq x_r+k-s(x_r)$.

Proof If $x_l + k \leq x_r - k$ then it is clear. Otherwise $x_l + k > x_l - k$. Assume for the contradiction that $x_l + k - s(x_l) > x_r + k - s(x_r)$ holds. Since $s(x_r) \neq -1$, a (k, r)-gathering A_r of $C(x_r, s(x_r))$ to $F(x_r)$ exists. Modify A_r so that the customers assigned to x_r to be assigned to x_l. The resulting assignment is a (k, r)-gathering of $C(x_l, s')$ to $F(x_l)$ with some $s' > s(x_l)$. A contradiction. □

Lemma 6 $s(x_i) \neq -1$ if and only if
either
(i) $|x_i - c_{min}| \leq k$,
(ii) $|x_i - c_{min}| > k$ and $(x_c + k, x_i - k)$ has no customer
where x_c is the facility having the maximum coordinate satisfying $x_c + k < x_i - k$ and $s(x_c) \neq -1$,
(iii) $|x_i - c_{min}| > k$, $x_c + k > x_i - k$, $x_c + k - s(x_c) < x_i - k$, and $[x_i - k, x_i + k]$ contains r or more customers
where x_c is the facility having the minimum coordinate satisfying $x_c + k \geq x_i - k$ and $s(x_c) \neq -1$, or
(iv) $|x_i - c_{min}| > k$, $x_c + k - s(x_c) \geq x_i - k$ and $(x_c + k - s(x_c), x_i + k]$ contains r or more customers
where x_c is the facility having the minimum coordinate satisfying $x_c + k \geq x_i - k$ and $s(x_c) \neq -1$.

Proof Assume $s(x_i) \neq -1$ then PG(x_i) has some solution. Then PG(x_i) has a non-overlapping solution A by Lemma 1. We have two cases. If all customers are assigned to x_i in A then (i) holds. Otherwise A has two or more open facilities and $|x_i - c_{min}| > k$ holds. Let x'_c be the open facility in A having the maximum coordinate except x_i. If $x'_c + k < x_i - k$ then, since $(x'_c + k, x_i - k)$ has no customer and $x_c \geq x'_c$, (ii) holds. Otherwise $x'_c + k \geq x_i - k$ holds. If $x'_c + k - s(x'_c) < x_i - k$ then, since $x_c \leq x'_c$ and Lemma 5 means $x_c + k - s(x_c) \leq x'_c + k - s(x'_c)$, (iii) holds. Otherwise $x'_c + k - s(x'_c) \geq x_i - k$ then similarly either (iii) or (iv) holds.

Assume either (i), (ii), (iii) or (iv) holds. If (i) holds then PG(x_i) has a simple (k, r)-gathering, in which all customers are assigned to x_i. If either (ii), (iii) or (iv) holds then by assigning the customers in either $[x_i - k, x_i + k]$ or $(x_c + k - s(x_c), x_i + k]$ to x_i we can extend a solution of PG(x_c) to a solution of PG(x_i). □

Now by Lemma 6 we can skip many parts of **Algorithm find-gathering2**. We have the following algorithm.

Algorithm find-gathering3(C, F, k, r)
01. let c_{min} and c_{max} be the minimum and the maximum in C
02. let c_r be the r-th smallest coordinate in C
03. **for** each facility $x_i \in F$ with $|x_i - c_{min}| \le k$ (in increasing order) /* condition (i) */
04. /* $[x_i - k, x_i + k]$ contains r or more customers */
05. $s(x_i) = x_i + k - c_r$
06. **for** each facility $x_i \in F$ with $|x_i - c_{min}| > k$ (in increasing order)
07. **begin**
08. $s(x_i) = -1$
09. **if** there is a facility x_c satisfying $x_c + k < x_i - k$ and $s(x_c) \ne -1$ **then** /* condition (ii) */
10. **begin**
11. let x_c be the maximum of such facilities
12. **if** $(x_c + k, x_i - k)$ has no customer
13. **then** $s(x_i) = x_i + k - c'$ where c' is the r-th smallest customer in $[x_i - k, x_i + k]$
14. **end**
15. **if** there is a facility x_c satisfying $x_c + k \ge x_i - k$ and $s(x_c) \ne -1$ **then** /* condition(iii), (iv) */
16. **begin**
17. let x_c be the minimum of such facilities
18. **if** both $(x_c + k - s(x_c), x_i + k]$ and $[x_i - k, x_i + k]$ has r or more customers **then**
19. **if** $x_c + k - s(x_c) \ge x_i - k$
20. **then** $s(x_i) = x_i + k - c''$ where c'' is the r-th smallest customer in $(x_c + k - s(x_c), x_i + k]$
21. **else** $s(x_i) = x_i + k - c''$ where c'' is the r-th smallest customer in $[x_i - k, x_i + k]$
22. **end**
23. **end**
24. Ans = "not exist"
25. **for** each facility $x_i \in F$ with $|x_i - c_{max}| \le k$
26. **if** $s(x_i) \ge 0$
27. **then** Ans = "exists"
28. **return** Ans

To check the condition (ii) of Lemma 6 efficiently we maintain x_c for the current x_i so that x_c is the facility having the maximum x_c satisfying (1) $x_c + k < x_i - k$ and (2) $s(x_c) \ne -1$. We need $O(|C| + |F|)$ time in total for this maintenance. Also to find x_c in line 17 efficiently we maintain the list L of "useful facilities" for the current x_i so that L contains all facilities x_j satisfying (1) $x_j + k \ge x_i - k$, (2) $s(x_j) \ne -1$. Then we can find x_c in line 17 in constant time. We need $O(|C| + |F|)$ time in total for this maintenance. Thus we have the following theorem.

Theorem 1 One can solve the (k, r)-gathering problem in $O(|C| + |F|)$ time if all customers and facilities are on a line.

By a simple dynamic programming algorithm on array $s(x_i)$ one can compute a (k, r)-gathering with the minimum number of open facilities.

5 Conclusion

In this paper we designed a linear time algorithm to solve the (k, r)-gathering problem when the customers and facilities are on a line.

If each customer has a weight, and the sum of the weights of the customers assigned to each open facility should be r or more, then it is the weighted version of the (k, r)-gathering problem. Unfortunately it is NP-complete even for $|F| = 2$, as follows. Given a set S of integers, problem PARTITION asks for the existance of a subset S' with $\sum_{a \in S'} a = \sum_{a \notin S'} a$. PARTITION is NP-complete [2]. We can transform PARTITION to the weighted version of the (k, r)-gathering problem as follows. Let $S = \{a_1, a_2, \ldots, a_n\}$. Locate each customer with weight a_i at i for each $i = 1, 2, \ldots, n$. Set $F = \{1, n\}$, $k = n$ and $r = (\sum_{i=1}^{n} a_i)/2$. Now a (k, r)-gathering exists if and only if PARTITION has a solution.

If customers can gather at any place, not restricted at potential facility locations, then one can perturbate any solution of the problem so that every gathering place is at $c_i + k$ for some c_i. So, by locating possible facilities at $c_i + k$ for each $c_i \in C$, we can find a solution of this problem using our algorithm. The running time is $O(|C| + |F|) = O(|C| + |C|) = O(|C|)$.

References

[1] A. Armon, *On min-max r-gatherings*, Theoretical Computer Science, 412, 573-582 (2011).

[2] M. R. Garey and D. S. Johnson, *Computers and Intractability: a Guide to the Theory of NP-Completeness*, Freeman, San Francisco (1979).