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A Discrete Mechanics Approach to Vibration Suppression Control of Free-fixed Euler-Bernoulli Beams

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Abstract—This research develops a new control method for free-fixed Euler-Bernoulli beams based on a blending method of discrete mechanics and nonlinear optimization. First, discrete Euler-Lagrange equations and discrete boundary equations are derived under the assumption on free boundary conditions. Next, for a discrete mechanics model with control inputs, a finite-dimensional nonlinear optimal control problem is formulated by setting an objective function. Then, the proposed control method is applied to a vibration suppression problem of free-fixed Euler-Bernoulli beams, and it turns out that the vibration of the beam is suppressed and the whole of the beam is stabilized at the desired time from the results of numerical simulations.

1. INTRODUCTION

It is well known that control of distributed parameter systems is far more difficult than control of concentrated parameter systems. Various studies on control of distributed parameter systems have been done in the past, and there exist two approaches to the research: the analytic approach and the numerical one. For the numerical approach, the authors have developed a new method called "discrete mechanics" by extending the existing method for concentrated parameter systems [1, 2, 3, 4, 5] to distributed parameter systems in [6, 7, 8]. In addition, we have derived discrete mechanics under free boundary conditions and applied the results to vibration suppression control of a free-fixed string [9].

The aim of this study is to develop a new control method based on discrete mechanics for the vibration suppression control problem of a free-fixed Euler-Bernoulli Beam, which is harder to control than a free-fixed string. First, Section 2 derives discrete mechanics for distributed parameter mechanical systems under free boundary conditions for the case where a continuous Lagrangian density includes through second-order partial derivative the displacement variable. Next, Section 3 proposes a new control method based on a blending method of discrete mechanics and nonlinear optimization. Then, in Section 4, we apply the new control method to the vibration suppression control problem of a free-fixed Euler-Bernoulli Beam, and a numerical simulation is performed to confirm the effectiveness of the new method.

2. DISCRETE MECHANICS UNDER FREE BOUNDARY CONDITIONS

First, in this section, the theory of discrete mechanics under free boundary conditions will be derived. As shown in Fig. 1, let us denote the time variable as $t \in \mathbf{R}$ and in the position of the 1-dimensional space as $x \in \mathbf{R}$. We also refer a displacement of the system at the time t and the position x as $u(t,x) \in \mathbf{R}$, and $u(t,x)$ with a subscript indicates partial derivative of $u(t,x)$ with respect to the subscript, e.g. u_t, u_x, u_{tt}, u_{tx}, u_{xx}. In this paper, we deal with a continuous Lagrangian density which includes through second-order partial derivative of $u(t,x)$ as $L^c(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx})$. Next, we consider discretization of variables. The time variable t and the position x are discretized with sampling intervals h and d as $t \approx hk \ \ (k = 1,2,\cdots,K)$, $\ x \approx dl \ \ (l = 1,2,\cdots,L)$, where $k \in \mathbf{Z}\ (1 \le k \le K)$ and $l \in \mathbf{Z}\ (1 \le l \le L)$ are indices of t and x, respectively. Now, we use a new notation $U_{k,l} \in \mathbf{R}$ as a discrete version of the displacement of the system at the time step k and the position l. Then, as shown in Fig. 1, the displacement of the system at the time t and the position x: $u(t,x)$ is represented as

$$\begin{aligned} u(t,x) \approx (1-\alpha)(1-\beta)U_{k,l} + (1-\alpha)\beta U_{k,l+1} \\ + \alpha(1-\beta)U_{k+1,l} + \alpha\beta U_{k+1,l+1} \end{aligned} \tag{1}$$

with four data $U_{k,l}$, $U_{k,l+1}$, $U_{k+1,l}$, $U_{k+1,l+1}$, where $\alpha, \beta \in \mathbf{R}$ are dividing parameters ($0 < \alpha,\ \beta < 1$). Partial derivatives of $u(t,x)$ are also represented by

$$\begin{aligned} u_t(t,x) &\approx \frac{U_{k+1,l} - U_{k,l}}{h}, \\ u_x(t,x) &\approx \frac{U_{k,l+1} - U_{k,l}}{d}, \\ u_{tt}(t,x) &\approx \frac{U_{k+1,l} - 2U_{k+1,l} + U_{k-1,l}}{h^2}, \\ u_{tx}(t,x) &\approx \frac{U_{k+1,l+1} - U_{k,l+1} - U_{k+1,l} + U_{k,l}}{hd}, \\ u_{xx}(t,x) &\approx \frac{U_{k,l+1} - 2U_{k,l} + U_{k,l-1}}{d^2}. \end{aligned} \tag{2}$$

Substituting the discretizing settings to L^c and multiplying hd to it, we define "a discrete Lagrangian density" as

$L^d(k,l,U_{k-1,l},U_{k,l-1},U_{k,l},U_{k,l+1},U_{k+1,l},U_{k+1,l+1})$. Then, by using discrete Hamilton's principle under free boundary conditions, we can derive "discrete Euler-Lagrange equations" and "discrete boundary equations" as follows.

Theorem 1 : For the discrete Lagrangian density $L^d_{k,l}$, *discrete Euler-Lagrange equations* that satisfy discrete Hamilton's principle are given by

$$\begin{aligned}&\frac{\partial L^d_{k-1,l-1}}{\partial U_{k,l}}+\frac{\partial L^d_{k-1,l}}{\partial U_{k,l}}+\frac{\partial L^d_{k,l-1}}{\partial U_{k,l}}\\&\quad+\frac{\partial L^d_{k,l}}{\partial U_{k,l}}+\frac{\partial L^d_{k,l+1}}{\partial U_{k,l}}+\frac{\partial L^d_{k+1,l}}{\partial U_{k,l}}=0\\&\quad(k=3,\cdots,K-2;\ \ l=3,\cdots,L-2),\end{aligned}\tag{3}$$

and *discrete boundary equations* are given by

$$\begin{aligned}&\frac{\partial L^d_{k-1,L-2}}{\partial U_{k,L-1}}+\frac{\partial L^d_{k-1,L-1}}{\partial U_{k,L-1}}+\frac{\partial L^d_{k,L-2}}{\partial U_{k,L-1}}\\&\quad+\frac{\partial L^d_{k,L-1}}{\partial U_{k,L-1}}+\frac{\partial L^d_{k+1,L-1}}{\partial U_{k,L-1}}=0\\&\quad(k=3,4,\cdots,K-2),\end{aligned}\tag{4}$$

$$\frac{\partial L^d_{k-1,L-1}}{\partial U_{k,L}}+\frac{\partial L^d_{k,L-1}}{\partial U_{k,L}}=0\quad(k=3,\cdots,K-2).\tag{5}$$

(Outline of Proof) Define *the discrete action sum* as

$$S^d(U):=\sum_{k=1}^{K-1}\sum_{l=1}^{L-1}L^d_{k,l},\tag{6}$$

and consider *discrete variations* of (6):

$$\delta S^d(U):=S^d(U+\delta U)-S^d(U),\tag{7}$$

where δU is a variation of U and satisfies the boundary conditions:

$$\delta U_{1,l}=\delta U_{2,l}=\delta U_{K-1,l}=\delta U_{K,l}=0\ \ (l=1,\cdots,L),\tag{8}$$

$$\delta U_{k,1}=\delta U_{k,2}=0\ \ (k=3,\cdots,K-2),\tag{9}$$

$$\delta U_{k,L-1}=\delta U_{k,L}\neq 0\ \ (k=3,\cdots,K-2),\tag{10}$$

where (10) is the free boundary conditions. Calculating (7) in detail, we have

$$\begin{aligned}&\delta S^d(U)\\&=\sum_{k=3}^{K-2}\sum_{l=3}^{L-2}\left(\frac{\partial L^d_{k-1,l-1}}{\partial U_{k,l}}+\frac{\partial L^d_{k-1,l}}{\partial U_{k,l}}+\frac{\partial L^d_{k,l-1}}{\partial U_{k,l}}\right.\\&\qquad\left.+\frac{\partial L^d_{k,l}}{\partial U_{k,l}}+\frac{\partial L^d_{k,l+1}}{\partial U_{k,l}}+\frac{\partial L^d_{k+1,l}}{\partial U_{k,l}}\right)\delta U_{k,l}\\&+\sum_{k=3}^{K-2}\left(\frac{\partial L^d_{k-1,L-2}}{\partial U_{k,L-1}}+\frac{\partial L^d_{k-1,L-1}}{\partial U_{k,L-1}}+\frac{\partial L^d_{k,L-2}}{\partial U_{k,L-1}}\right.\\&\qquad\left.+\frac{\partial L^d_{k,L-1}}{\partial U_{k,L-1}}+\frac{\partial L^d_{k+1,L-1}}{\partial U_{k,L-1}}\right)\delta U_{k,L-1}\\&+\sum_{k=3}^{K-2}\left(\frac{\partial L^d_{k,L-1}}{\partial U_{k,L-1}}+\frac{\partial L^d_{k+1,L-1}}{\partial U_{k,L-1}}\right)\delta U_{k,L}\end{aligned}\tag{11}$$

Using "discrete Hamilton's principle" [6, 7, 9] for (11), we obtain the discrete Euler-Lagrange equations (3) and the discrete boundary conditions (4), (5).

Note that the discrete Euler-Lagrange equations (3) are represented as a set of difference equations that include up to 17 variables: $U_{k-2,l-1}$, $U_{k-2,l}$, $U_{k-1,l-2}$, $U_{k-1,l-1}$, $U_{k-1,l}$, $U_{k-1,l+1}$, $U_{k,l-2}$, $U_{k,l-1}$, $U_{k,l+1}$, $U_{k,l+1}$, $U_{k,l+2}$, $U_{k+1,l-1}$, $U_{k+1,l}$, $U_{k+1,l+1}$, $U_{k+1,l+2}$, $U_{k+2,l}$, $U_{k+2,l+1}$, and discrete boundary equations (4) and (5) are represented as a set of difference equations that include up to 15 variables: $U_{k-2,L-2}$, $U_{k-2,L-1}$, $U_{k-1,L-3}$, $U_{k-1,L-2}$, $U_{k-1,L-1}$, $U_{k-1,L}$, $U_{k,L-3}$, $U_{k,L-2}$, $U_{k,L-1}$, $U_{k,L}$, $U_{k+1,L-2}$, $U_{k+1,L-1}$, $U_{k+1,L}$, $U_{k+2,L-1}$, $U_{k+2,L}$ and up to 9 variables: $U_{k-2,L-1}$, $U_{k-1,L-2}$, $U_{k-1,L-1}$, $U_{k-1,L}$, $U_{k,L-2}$, $U_{k,L-1}$, $U_{k,L}$, $U_{k+1,L-1}$, $U_{k+1,L}$, respectively. All the KL displacements $U_{k,l}$ $(1\leq k\leq K;\ \ 1\leq l\leq L)$ can be calculated by (3)–(5). under suitable initial and boundary conditions. In addition, the discrete Euler-Lagrange equations (3) and the discrete boundary equations (4), (5) are generally nonlinear and implicit, and hence we need some numerical solutions for nonlinear equations such as Newton's method in order to calculate all the displacements of the system.

3. OPTIMAL CONTROL METHOD BASED ON DISCRETE MECHANICS

In this section, a nonlinear control problem for a mathematical model derived by discrete mechanics is formulated, and a solution method of the problem is considered. First, the setting on control inputs is shown. Denote a control input at the time step k and the position l as $F_{k,l}\in\mathbf{R}$. If an actuator is not installed at the position l, we set $F_{k,l}\equiv 0$ $(k=1,\cdots,K)$. We also denote and a set of indices l such that actuators are installed as Δ. Thus, the discrete Euler-Lagrange equations with control inputs are given by

$$\begin{aligned}&\frac{\partial L^d_{k-1,l-1}}{\partial U_{k,l}}+\frac{\partial L^d_{k-1,l}}{\partial U_{k,l}}+\frac{\partial L^d_{k,l-1}}{\partial U_{k,l}}\\&\quad+\frac{\partial L^d_{k,l}}{\partial U_{k,l}}+\frac{\partial L^d_{k,l+1}}{\partial U_{k,l}}+\frac{\partial L^d_{k+1,l}}{\partial U_{k,l}}=F_{k,l}.\\&(k=3,4,\cdots,K-2;\ \ l=3,4,\cdots,L-2)\end{aligned}\tag{12}$$

In this study, we deal with the next control problem for the discrete mechanics model.

Problem 1 : For the discrete Lagrangian density, the discrete Euler-Lagrange equation with control inputs (12), and the discrete boundary equations (4), (5), find control inputs $F_{k,l}$ $(k=2,\cdots,K-1;\ \ l\in\varDelta)$ that make all the specified displacements $U_{k,l}$ $(k=\kappa,\cdots,K;\ \ l=1,\cdots,L)$ converge to 0.

In order to solve Problem 1, we consider an optimal control approach. Using weight parameters a, b, c, we set an evaluation function as

$$J(U,F) = a\sum_{k=1}^{\kappa-1}\sum_{l=1}^{L} U_{k,l}^2 + b\sum_{k=\kappa}^{K}\sum_{l=1}^{L} U_{k,l}^2 + c\sum_{k=3}^{K-2}\sum_{l\in\Delta} F_{k,l}^2, \tag{13}$$

where the first and second terms evaluate the displacements from $k = 1$ to $k = \kappa - 1$ and ones from $k = \kappa$ to $k = K$, respectively, and the third term evaluates the values of control inputs for all time steps. In (13), κ is a design parameter called *a stabilization start time step*. It can be expect that we can make all the specified displacements converge to 0. by minimizing the evaluation function (13). The optimal control problem for the discrete Euler-Lagrange equation with control inputs (12) can be formulated as

$$\begin{aligned} &\min_{U,F}\ (13),\\ &\text{subject to}\ (12),\ (4),\ (5),\\ &\text{given}\ \text{initial conditions},\ \text{boundary conditions}. \end{aligned} \tag{14}$$

The optimal control problem (14) can be referred as a finite-dimensional nonlinear optimization problem with constraints, and hence we can solve it by numerical solutions such as the sequential quadratic programming method [10]. It is known that the sequential quadratic programming method can be applied to a relatively large-scale problems and effectively obtain an optimal or near-optimal solution.

4. SIMULATIONS

This section considers the vibration suppression control of a free-fixed Euler-Bernoulli beam based on the new control method as shown in Section 3. We here consider the Euler-Bernoulli beam clamped at both ends as illustrated in Fig. 1. Denote the position of the beam as x and the displacement of the beam at time t and the position x as $u(t, x)$. The physical parameters of the beam are set as ρ: a energy density of the beam, μ: bending stiffness of the beam. Then, the continuous Lagrangian density of the beam is given by

$$L^c = \frac{1}{2}\rho u_t^2 - \frac{1}{2}\mu u_{xx}^2 \tag{15}$$

Note that the continuous Lagrangian density (15) contains not only a first order partial derivative u_t but also a second order partial derivative u_{xx}. From (15), we have the discrete Lagrangian density:

$$L_{k,l}^d = \frac{1}{2}\rho hd\left(\frac{U_{k+1,l} - U_{k,l}}{h}\right)^2 - \frac{1}{2}\mu hd\left(\frac{U_{k,l+1} - 2U_{k,l} + U_{k,l-1}}{d^2}\right)^2, \tag{16}$$

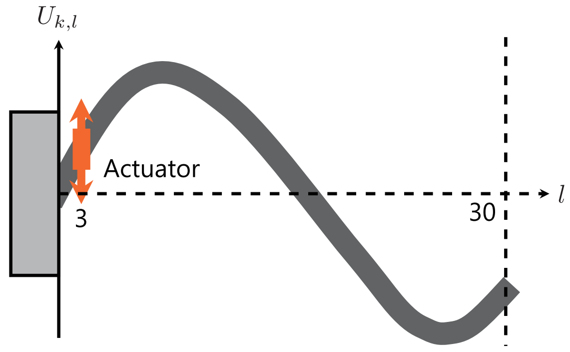

Figure 1: Setting of Numerical Simulation

and hence from (3) we obtain the discrete Euler-Lagrange equation of the beam as

$$-\frac{\mu}{h^2}U_{k-1,l} - \frac{\mu}{d^4}U_{k,l-2} + \frac{4\mu}{d^4}U_{k,l-1} + \left(\frac{2\rho}{h^2} - \frac{6\mu}{d^4}\right)U_{k,l} + \frac{4\mu}{d^4}U_{k,l+1} - \frac{\mu}{d^4}U_{k,l+2} - \frac{\rho}{h^2}U_{k+1,l} = 0, \tag{17}$$

where (17) contains 7 displacement variables $U_{k-1,l}$, $U_{k,l-2}$, $U_{k,l-1}$, $U_{k,l}$, $U_{k,l+1}$, $U_{k,l+2}$, $U_{k,l+1}$. In addition, we also have the discrete boundary equations from (4), (5) as

$$-\frac{\rho d}{h}U_{k-1,L-1} - \frac{\mu h}{d^3}U_{k,L-3} + \frac{4\mu h}{d^3}U_{k,L-2} + \left(\frac{\rho d}{h} - \frac{5\mu h}{d^3}\right)U_{k,L-1} + \frac{2\mu h}{d^3}U_{k,L} - \frac{\rho d}{h}U_{k+1,L-1} = 0, \tag{18}$$

$$U_{k,L-2} - 2U_{k,L-1} + U_{k,L} = 0, \tag{19}$$

where (18) contains 6 displacement variables $U_{k-1,L-1}$, $U_{k,L-3}$, $U_{k,L-2}$, $U_{k,L-1}$, $U_{k,L}$, $U_{k+1,L-1}$ and (19) contains 3 displacement variables $U_{k,L-2}$, $U_{k,L-1}$, $U_{k,L}$.

Then, numerical simulations are carried out by the proposed control method in order to check the effectiveness. We assume that the number of control inputs is 1, that is to say, the actuator that can generate a control input is installed at only the extreme left of the beam as illustrated in Fig. 1. The parameters are set as the physical parameters: $\rho = 1$, $\mu = 0.1$, the sampling intervals: $h = 0.01$, $d = 0.1$, the total steps: $K = 300$, $L = 30$, the set of actuator indices: $\Delta = \{3\}$, the stabilization start time step: $\kappa = 250$, the weight parameters of evaluation function: $a = 1$, $b = 50000$, $c = 1$.

Figs. 2 and 3 show simulation results. Fig. 2 shows a 3D plot of the displacements of the beam $U_{k,l}$, and fig. 3 illustrates a time history on average of the absolute value of $U_{k,l}$:

$$\frac{1}{L}\sum_{l=1}^{L}|U_{k,l}|. \tag{20}$$

From these results, it can be confirmed that all the displacements of the beam in the desired time step $k = 250 - 300$

converge to 0, and hence vibration suppression control is achieved. Consequently, the simulation result shows the effectiveness of the new control method.

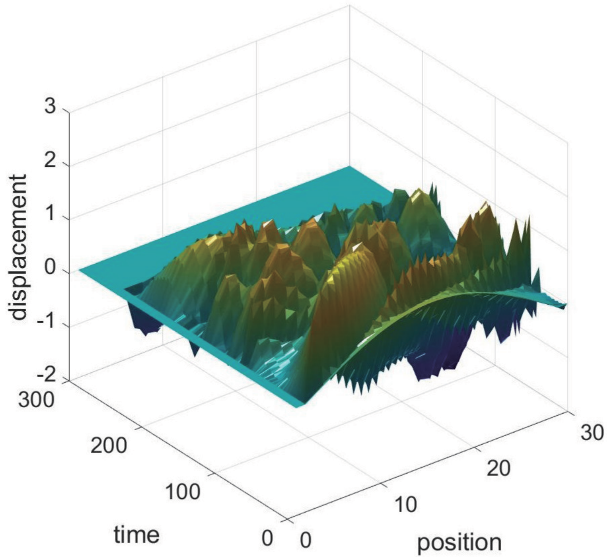

Figure 2: 3D Plot of Displacement of Beam

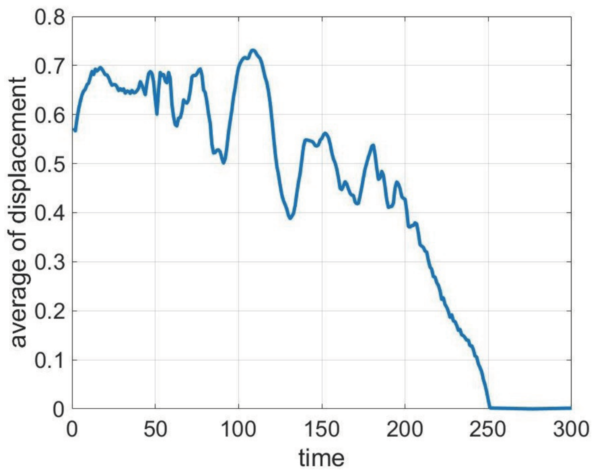

Figure 3: Time History of Average of Displacement

5. CONCLUSIONS

This study has developed a new control method for a free-fixed Euler-Bernoulli beam based on discrete mechanics. From simulation results, it turns out that vibration of the free-fixed Euler-Bernoulli beam is suppressed and the whole of the beam is stabilized by the new method. Hence, we can say that the new control method is effective for control of distributed parameter mechanical systems.

The future work includes the following topics; theoretical analysis on discrete Euler-Lagrange equations, Extension to multi-dimensional distributed parameter mechanical systems, and development of model-predictive feedback control methods.

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