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Improvement of Sinc-Nyström Methods for Initial Value Problems

Ryota Hara† and Tomoaki Okayama†

†Graduate School of Information Sciences, Hiroshima City University
3–4–1, Ozuka-higashi, Asaminami-ku, Hiroshima 731-3194, Japan
Email: mc68011@e.hiroshima-cu.ac.jp, okayama@hiroshima-cu.ac.jp

Abstract—Nurmuhammad et al. proposed two Sinc-Nyström methods for initial value problems. The difference between the two methods lies in the variable transformations employed in them. One method uses a single-exponential (SE) transformation and the other uses a double-exponential (DE) transformation. This paper proposes new methods by replacing these transformations to achieve a high convergence rate for the method with the SE transformation, and to enable the inverse function of the DE transformation to be written with elementary functions.

1. Introduction

The focus of this study is on the numerical methods used for initial value problems of the following form:

$$\begin{cases} \boldsymbol{y}'(t) = K(t)\boldsymbol{y}(t) + \boldsymbol{g}(t), \\ \boldsymbol{y}(0) = \boldsymbol{r}, \end{cases} \tag{1}$$

where $K(t)$ is an $m \times m$ matrix whose (i, j) component is $k_{ij}(t)$, and $\boldsymbol{y}(t)$, $\boldsymbol{g}(t)$, and $\boldsymbol{r}$ are m-dimensional vectors. In this paper, the solution $\boldsymbol{y}(t)$ is supposed to decay exponentially as $t \to \infty$. For such a case, Nurmuhammad et al. [1] proposed two types of Sinc-Nyström methods, which were developed based on the Sinc indefinite integration and a variable transformation. A single-exponential (SE) transformation is employed in one method, and we call this method the SE-Sinc-Nyström method. The other method employs a double-exponential (DE) transformation, and we call it the DE-Sinc-Nyström method. They reported that these methods can attain exponential convergence even for stiff problems.

This study improves their methods by replacing the variable transformations (a) to achieve a higher convergence rate for the SE-Sinc-Nyström method and (b) to permit the inverse function of the DE transformation to be written with elementary functions for the DE-Sinc-Nyström method.

The remainder of this paper is organized as follows: the Sinc indefinite integration combined with a variable transformation is described in Section 2, the Sinc-Nyström methods developed by Nurmuhammad et al. are described in Section 3, the improved Sinc-Nyström methods by this study are presented in Section 4, numerical examples are shown in Section 5, and conclusions are given in Section 6.

2. Sinc Indefinite Integration and Variable Transformations

The Sinc indefinite integration over the whole real axis is expressed as

$$\int_{-\infty}^{\xi} F(x)\,\mathrm{d}x \approx \sum_{j=-M}^{N} F(jh)J(j,h)(\xi), \quad \xi \in \mathbb{R}, \tag{2}$$

where $J(j,h)(x)$ is defined by using the sine integral $\mathrm{Si}(x) = \int_0^x \{(\sin t)/t\}\,\mathrm{d}t$ as

$$J(j,h)(x) = h\left\{\frac{1}{2} + \frac{1}{\pi}\,\mathrm{Si}\left(\frac{\pi(x - jh)}{h}\right)\right\}.$$

In what follows, we consider the application of the Sinc indefinite integration to

$$I(t) := \int_0^t f(s)\,\mathrm{d}s, \quad t \in (0,\infty).$$

For this purpose, we should employ a proper variable transformation to transform $I(t)$ to the left-hand side in (2).

2.1. SE Transformations

Stenger [2] considered an SE transformation

$$t = \psi_{\mathrm{N}}(x) = \operatorname{arcsinh}(\mathrm{e}^x),$$

and derived the following formula:

$$\begin{aligned} I(t) &= \int_{-\infty}^{\psi_{\mathrm{N}}^{-1}(t)} f(\psi_{\mathrm{N}}(x))\psi_{\mathrm{N}}'(x)\,\mathrm{d}x \\ &\approx \sum_{j=-M}^{N} f(\psi_{\mathrm{N}}(jh))\psi_{\mathrm{N}}'(jh)J(j,h)(\psi_{\mathrm{N}}^{-1}(t)). \end{aligned} \tag{3}$$

Instead of $\psi_{\mathrm{N}}(x)$, Muhammad–Mori [3] considered another SE transformation

$$t = \psi_{\mathrm{H}}(x) = \log(1 + \mathrm{e}^x),$$

and derived the following formula:

$$\begin{aligned} I(t) &= \int_{-\infty}^{\psi_{\mathrm{H}}^{-1}(t)} f(\psi_{\mathrm{H}}(x))\psi_{\mathrm{H}}'(x)\,\mathrm{d}x \\ &\approx \sum_{j=-M}^{N} f(\psi_{\mathrm{H}}(jh))\psi_{\mathrm{H}}'(jh)J(j,h)(\psi_{\mathrm{H}}^{-1}(t)). \end{aligned} \tag{4}$$

Error analyses for the formulas (3) and (4) have already been given as follows. Here, let $\mathscr{D}_d$ denote a strip complex domain defined as $\mathscr{D}_d = \{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| < d\}$ for $d > 0$.

Theorem 1 (Okayama [4, Theorem 3.2]) *Assume that f is analytic in $\psi_{\mathrm{N}}(\mathscr{D}_d)$ with $0 < d < \pi/2$ and that there exist positive constants K, α, and β such that*

$$|f(z)| \leq K \left|\frac{z}{1+z}\right|^{\alpha-1} |\mathrm{e}^{-z}|^{\beta} \tag{5}$$

holds for all $z \in \psi_{\mathrm{N}}(\mathscr{D}_d)$. Let $\mu = \min\{\alpha, \beta\}$, let M and N be defined as

$$\begin{cases} M = n, \quad N = \lceil \alpha n/\beta \rceil & (\text{if } \mu = \alpha), \\ N = n, \quad M = \lceil \beta n/\alpha \rceil & (\text{if } \mu = \beta), \end{cases} \tag{6}$$

and let h be defined as

$$h = \sqrt{\frac{\pi d}{\mu n}}. \tag{7}$$

Then, there exists a positive constant C independent of n such that

$$\sup_{t\in(0,\infty)} \left| I(t) - \sum_{j=-M}^{N} f(\psi_{\mathrm{N}}(jh))\psi_{\mathrm{N}}'(jh) J(j,h)(\psi_{\mathrm{N}}^{-1}(t)) \right| \leq C\,\mathrm{e}^{-\sqrt{\pi d \mu n}}. \tag{8}$$

Theorem 2 (Hara–Okayama [5, Theorem 2]) *Assume that f is analytic in $\psi_{\mathrm{H}}(\mathscr{D}_d)$ with $0 < d < \pi$ and that there exist positive constants K, β, and α with $0 < \alpha \leq 1$ such that* (5) *holds for all $z \in \psi_{\mathrm{H}}(\mathscr{D}_d)$. Let $\mu = \min\{\alpha, \beta\}$, let M and N be defined as in* (6)*, and let h be defined as in* (7)*. Then, there exists a positive constant C independent of n such that*

$$\sup_{t\in(0,\infty)} \left| I(t) - \sum_{j=-M}^{N} f(\psi_{\mathrm{H}}(jh))\psi_{\mathrm{H}}'(jh) J(j,h)(\psi_{\mathrm{H}}^{-1}(t)) \right| \leq C\,\mathrm{e}^{-\sqrt{\pi d \mu n}}. \tag{9}$$

In view of (8) and (9), the convergence rates of the two methods seem to be the same: $\mathrm{O}(\mathrm{e}^{-\sqrt{\pi d \mu n}})$. However, the rate in Theorem 2 may be higher than that in Theorem 1 in reality. This occurs because the value of d in Theorem 2 may be taken larger than that in Theorem 1.

2.2. DE Transformations

Instead of the SE transformations, Takahasi–Mori [6] proposed an improved variable transformation, the so-called DE transformation,

$$t = \phi_{\mathrm{N}}(x) = \exp(x - \mathrm{e}^{-x}).$$

Although they did not consider the indefinite integral $I(t)$ (but the definite integral $I(\infty)$), we can derive the following formula using the transformation as

$$\begin{aligned} I(t) &= \int_{-\infty}^{\phi_{\mathrm{N}}^{-1}(t)} f(\phi_{\mathrm{N}}(x))\phi_{\mathrm{N}}'(x)\,\mathrm{d}x \\ &\approx \sum_{j=-M}^{N} f(\phi_{\mathrm{N}}(jh))\phi_{\mathrm{N}}'(jh) J(j,h)(\phi_{\mathrm{N}}^{-1}(t)). \end{aligned}$$

However, we cannot express its inverse function $\phi_{\mathrm{N}}^{-1}(t)$ with elementary functions, and any error analysis for the formula has not been given. To remedy these issues, Okayama [4] proposed another DE transformation

$$t = \phi_{\mathrm{H}}(x) = \log(1 + \mathrm{e}^{\pi \sinh(x)}),$$

and derived the following formula:

$$\begin{aligned} I(t) &= \int_{-\infty}^{\phi_{\mathrm{H}}^{-1}(t)} f(\phi_{\mathrm{H}}(x))\phi_{\mathrm{H}}'(x)\,\mathrm{d}x \\ &\approx \sum_{j=-M}^{N} f(\phi_{\mathrm{H}}(jh))\phi_{\mathrm{H}}'(jh) J(j,h)(\phi_{\mathrm{H}}^{-1}(t)), \end{aligned}$$

and performed its error analysis as below.

Theorem 3 (Okayama [4, Theorem 3.5]) *Assume that f is analytic in $\phi_{\mathrm{H}}(\mathscr{D}_d)$ with $0 < d < \pi/2$ and that there exist positive constants K, β, and α with $0 < \alpha \leq 1$ such that* (5) *holds for all $z \in \phi_{\mathrm{H}}(\mathscr{D}_d)$. Let $\mu = \min\{\alpha, \beta\}$, let M and N be defined as*

$$\begin{cases} M = n, \quad N = n - \lfloor (1/h) \cdot \log(\beta/\alpha) \rfloor & (\text{if } \mu = \alpha), \\ N = n, \quad M = n - \lfloor (1/h) \cdot \log(\alpha/\beta) \rfloor & (\text{if } \mu = \beta), \end{cases} \tag{10}$$

and let h be defined as

$$h = \frac{\log(2dn/\mu)}{n}.$$

Then, there exists a positive constant C independent of n such that

$$\sup_{t\in(0,\infty)} \left| I(t) - \sum_{j=-M}^{N} f(\phi_{\mathrm{H}}(jh))\phi_{\mathrm{H}}'(jh) J(j,h)(\phi_{\mathrm{H}}^{-1}(t)) \right| \leq C \frac{\log(2dn/\mu)}{n} \mathrm{e}^{-\pi d n/\log(2dn/\mu)}. \tag{11}$$

The convergence rate of (11) is much higher than that of (8) or (9).

3. Sinc-Nyström Methods by Nurmuhammad et al. [1]

3.1. SE-Sinc-Nyström Method

Integrating both sides of (1), we derive the following:

$$\boldsymbol{y}(t) = \boldsymbol{r} + \int_0^t \{K(s)\boldsymbol{y}(s) + \boldsymbol{g}(s)\}\,\mathrm{d}s. \tag{12}$$

Let $l = M + N + 1$ and let $\boldsymbol{y}^{(l)}(t)$ be an approximate solution of $\boldsymbol{y}(t)$. Approximating the integral in (12) based on Theorem 1, we derive

$$\boldsymbol{y}^{(l)}(t) = \boldsymbol{r} + \sum_{j=-M}^{N} \Big\{K(\psi_{\mathrm{N}}(jh))\boldsymbol{y}^{(l)}(\psi_{\mathrm{N}}(jh)) + \boldsymbol{g}(\psi_{\mathrm{N}}(jh))\Big\}\psi_{\mathrm{N}}'(jh)J(j,h)(\psi_{\mathrm{N}}^{-1}(t)). \quad (13)$$

To determine the unknown coefficients $\boldsymbol{y}^{(l)}(\psi_{\mathrm{N}}(jh))$, we set sampling points at $t = \psi_{\mathrm{N}}(ih)$ $(i = -M, -M+1, \ldots, N)$. Then, we obtain a system of linear equations given by

$$(I_m \otimes I_l - I_m \otimes \{hI_l^{(-1)}D_l^{(\psi_{\mathrm{N}})}\}[K_{ij}^{(\psi_{\mathrm{N}})}])\boldsymbol{Y}^{(\psi_{\mathrm{N}})} = \boldsymbol{R} + I_m \otimes \{hI_l^{(-1)}D_l^{(\psi_{\mathrm{N}})}\}\boldsymbol{G}^{(\psi_{\mathrm{N}})}, \quad (14)$$

where I_l and I_m are identity matrices, $\otimes$ denotes the Kronecker product, and $I_l^{(-1)}$ is an $l \times l$ matrix whose (i, j) components are defined as

$$(I_l^{(-1)})_{ij} = \frac{1}{2} + \frac{1}{\pi}\mathrm{Si}\,(\pi(i-j)), \quad (i, j = -M, -M+1, \ldots, N).$$

Moreover, $D_l^{(\psi_{\mathrm{N}})}$ and $K_{ij}^{(\psi_{\mathrm{N}})}$ are $l \times l$ diagonal matrices defined as

$$D_l^{(\psi_{\mathrm{N}})} = \mathrm{diag}[\psi_{\mathrm{N}}'(-Mh), \ldots, \psi_{\mathrm{N}}'(Nh)],$$
$$K_{ij}^{(\psi_{\mathrm{N}})} = \mathrm{diag}[k_{ij}(\psi_{\mathrm{N}}(-Mh)), \ldots, k_{ij}(\psi_{\mathrm{N}}(Nh))],$$

and $[K_{ij}^{(\psi_{\mathrm{N}})}]$ is a block matrix whose (i, j) component is $K_{ij}^{(\psi_{\mathrm{N}})}$ $(i, j = 1, \ldots, m)$. Furthermore, $\boldsymbol{R}$, $\boldsymbol{Y}^{(\psi_{\mathrm{N}})}$, and $\boldsymbol{G}^{(\psi_{\mathrm{N}})}$ are lm-dimensional vectors defined as follows:

$$\boldsymbol{R} = [r_1, \ldots, r_1, r_2, \ldots, r_2, \ldots, r_m, \ldots, r_m]^{\mathrm{T}},$$
$$\boldsymbol{Y}^{(\psi_{\mathrm{N}})} = [y_1^{(l)}(\psi_{\mathrm{N}}(-Mh)), \ldots, y_1^{(l)}(\psi_{\mathrm{N}}(Nh)), \ldots, y_m^{(l)}(\psi_{\mathrm{N}}(-Mh)), \ldots, y_m^{(l)}(\psi_{\mathrm{N}}(Nh))]^{\mathrm{T}},$$
$$\boldsymbol{G}^{(\psi_{\mathrm{N}})} = [g_1(\psi_{\mathrm{N}}(-Mh)), \ldots, g_1(\psi_{\mathrm{N}}(Nh)), \ldots, g_m(\psi_{\mathrm{N}}(-Mh)), \ldots, g_m(\psi_{\mathrm{N}}(Nh))]^{\mathrm{T}}.$$

By solving (14), we can get the value of $\boldsymbol{y}^{(l)}(\psi_{\mathrm{N}}(jh))$, from which $\boldsymbol{y}^{(l)}(t)$ is determined through (13). This procedure is the SE-Sinc-Nyström method proposed by Nurmuhammad et al.

3.2. DE-Sinc-Nyström Method

To accelerate the convergence rate, Nurmuhammad et al. [1] considered the replacement of ψ_{N} in the SE-Sinc-Nyström method by ϕ_{N}. Using this replacement in (13), we derive

$$\boldsymbol{y}^{(l)}(t) = \boldsymbol{r} + \sum_{j=-M}^{N} \Big\{K(\phi_{\mathrm{N}}(jh))\boldsymbol{y}^{(l)}(\phi_{\mathrm{N}}(jh)) + \boldsymbol{g}(\phi_{\mathrm{N}}(jh))\Big\}\phi_{\mathrm{N}}'(jh)J(j,h)(\phi_{\mathrm{N}}^{-1}(t)). \quad (15)$$

Herein, we suppose that they [1] set M and N according to (10), and set h as $h = \log(\pi dn/\mu)/n$. Setting sampling points at $t = \phi_{\mathrm{N}}(ih)$ $(i = -M, -M+1, \ldots, N)$, we obtain

$$(I_m \otimes I_l - I_m \otimes \{hI_l^{(-1)}D_l^{(\phi_{\mathrm{N}})}\}[K_{ij}^{(\phi_{\mathrm{N}})}])\boldsymbol{Y}^{(\phi_{\mathrm{N}})} = \boldsymbol{R} + I_m \otimes \{hI_l^{(-1)}D_l^{(\phi_{\mathrm{N}})}\}\boldsymbol{G}^{(\phi_{\mathrm{N}})}, \quad (16)$$

for which ϕ_{N} is used instead of ψ_{N}. By solving (16), we can get the value of $\boldsymbol{y}^{(l)}(\phi_{\mathrm{N}}(jh))$, from which $\boldsymbol{y}^{(l)}(t)$ is determined through (15). This procedure is the DE-Sinc-Nyström method proposed by Nurmuhammad et al.

4. Improved Sinc-Nyström Methods

4.1. SE-Sinc-Nyström Method

We consider an approximation of (12) based on Theorem 2. For this purpose, by replacing ψ_{N} with ψ_{H} in (13), we derive

$$\boldsymbol{y}^{(l)}(t) = \boldsymbol{r} + \sum_{j=-M}^{N} \Big\{K(\psi_{\mathrm{H}}(jh))\boldsymbol{y}^{(l)}(\psi_{\mathrm{H}}(jh)) + \boldsymbol{g}(\psi_{\mathrm{H}}(jh))\Big\}\psi_{\mathrm{H}}'(jh)J(j,h)(\psi_{\mathrm{H}}^{-1}(t)). \quad (17)$$

Moreover, by replacing ψ_{N} with ψ_{H} in (14), we obtain

$$(I_m \otimes I_l - I_m \otimes \{hI_l^{(-1)}D_l^{(\psi_{\mathrm{H}})}\}[K_{ij}^{(\psi_{\mathrm{H}})}])\boldsymbol{Y}^{(\psi_{\mathrm{H}})} = \boldsymbol{R} + I_m \otimes \{hI_l^{(-1)}D_l^{(\psi_{\mathrm{H}})}\}\boldsymbol{G}^{(\psi_{\mathrm{H}})}. \quad (18)$$

By solving (18), we can get the value of $\boldsymbol{y}^{(l)}(\psi_{\mathrm{H}}(jh))$, from which $\boldsymbol{y}^{(l)}(t)$ is determined through (17). This procedure is the SE-Sinc-Nyström method proposed by this study.

4.2. DE-Sinc-Nyström Method

We consider an approximation of (12) based on Theorem 3. For this purpose, by replacing ϕ_{N} with ϕ_{H} in (15), we derive

$$\boldsymbol{y}^{(l)}(t) = \boldsymbol{r} + \sum_{j=-M}^{N} \Big\{K(\phi_{\mathrm{H}}(jh))\boldsymbol{y}^{(l)}(\phi_{\mathrm{H}}(jh)) + \boldsymbol{g}(\phi_{\mathrm{H}}(jh))\Big\}\phi_{\mathrm{H}}'(jh)J(j,h)(\phi_{\mathrm{H}}^{-1}(t)). \quad (19)$$

Moreover, by replacing ϕ_{N} with ϕ_{H} in (16), we obtain

$$(I_m \otimes I_l - I_m \otimes \{hI_l^{(-1)}D_l^{(\phi_{\mathrm{H}})}\}[K_{ij}^{(\phi_{\mathrm{H}})}])\boldsymbol{Y}^{(\phi_{\mathrm{H}})} = \boldsymbol{R} + I_m \otimes \{hI_l^{(-1)}D_l^{(\phi_{\mathrm{H}})}\}\boldsymbol{G}^{(\phi_{\mathrm{H}})}. \quad (20)$$

By solving (20), we can get the value of $\boldsymbol{y}^{(l)}(\phi_{\mathrm{H}}(jh))$, from which $\boldsymbol{y}^{(l)}(t)$ is determined through (19). This procedure is the DE-Sinc-Nyström method proposed by this study.

5. Numerical Examples

In this section, we present numerical results for the following initial value problems (implemented in MATLAB R2016a on Windows 8 with Intel Xeon 2.50 GHz, 128GB memory).

Example 1 *Consider the following initial value problem:*

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} -2 & e^{-t} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \quad \begin{pmatrix} y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{21}$$

whose solution is $y(t) = t\,e^{-2t}$, $z(t) = e^{-t}$.

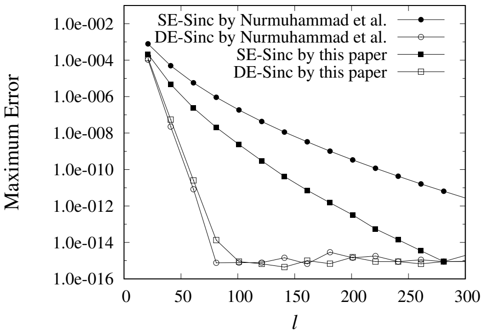

Figure 1: Errors of two SE-Sinc-Nyström methods and two DE-Sinc-Nyström methods for the problem (21).

Example 2 *Consider the following initial value problem:*

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \quad \begin{pmatrix} y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{22}$$

whose solution is $y(t) = e^{-2t}(\cos t + \sin t)$, $z(t) = y'(t)$.

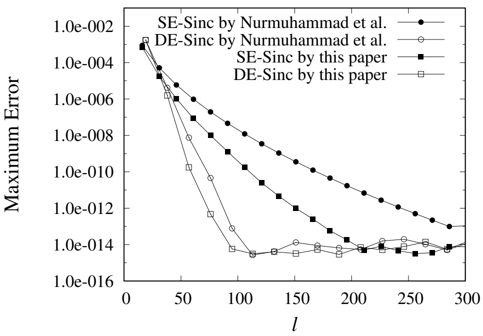

Figure 2: Errors of two SE-Sinc-Nyström methods and two DE-Sinc-Nyström methods for the problem (22).

For Example 1, we set $(\alpha, \beta, d) = (1, 1, 1.5)$ in the method using ψ_{N}, $(\alpha, \beta, d) = (1, 1, 3)$ in the method using ψ_{H}, and $(\alpha, \beta, d) = (1, 1, 1.5)$ in the method using ϕ_{N} or ϕ_{H}. For Example 2, we set $(\alpha, \beta, d) = (1, 2, 1.5)$ in the method using ψ_{N}, $(\alpha, \beta, d) = (1, 2, 3)$ in the method using ψ_{H}, and $(\alpha, \beta, d) = (1, 2, 1.5)$ in the method using ϕ_{N} or ϕ_{H}. The results for these problems are shown in Figures 1 and 2. The errors were investigated on the 101 points $t = 2^i$ $(i = -50, -49, \ldots, 49, 50)$, and the maximum error among these points was plotted. From the graphs, we see that the SE-Sinc-Nyström method by this study can converge faster than that by Nurmuhammad et al., and that the DE-Sinc-Nyström method by this study can achieve a similar convergence rate to that by them. Note that the method using ϕ_{N} is time-consuming because ϕ_{N}^{-1} cannot be written with elementary functions. In fact, for both examples, the method using ϕ_{N} required about 1.1-1.2 times the CPU computational time compared with that required by the method using ϕ_{H} at $l \simeq 100$.

6. Conclusions

To solve initial value problems, Nurmuhammad et al. [1] proposed the SE-Sinc-Nyström method and the DE-Sinc-Nyström method. The SE transformation $t = \psi_{\mathrm{N}}(x)$ is used in the former method and the DE transformation $t = \phi_{\mathrm{N}}(x)$ is used in the latter one. This study improved these methods by replacing ψ_{N} with ψ_{H} and ϕ_{N} with ϕ_{H}. With this replacement, the SE-Sinc-Nyström method can achieve a higher convergence rate, and the DE-Sinc-Nyström method becomes less time-consuming without compromising on the performance, which are observed by numerical examples.

Designing a better variable transformation will be one of our future pursuits. Another future pursuit would be to provide error analysis for the presented methods.

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