Coupling-enhanced deterministic stochastic resonance

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Abstract—We show that deterministic stochastic resonance (DSR) can be enhanced by coupling of chaotic oscillators and there is an optimal coupling strength which maximizes the resonance response. We study periodic-forced chaotic oscillators coupled to each other. The phases of the oscillators and periodic forcing synchronize with each other when the force strength is large. When we set the force strength below a critical value, the phase synchronization occasionally fails, and we can observe intermittent slips in the phase differences. The strength of the coupling between the oscillators is another bifurcation parameter that has a critical point between asynchronous and synchronous phase slip state. When the coupling is larger than the critical value, the DSR effect is enhanced by synchronized slips. Since the increasing of coupling strength also controls the average rate of slips, that is the internal fluctuation, we can see an optimal coupling strength which maximizes the DSR response. To our knowledge, this is the first report of enhanced DSR by coupling in a deterministic system.

1. Introduction

Noise-induced effects in nonlinear systems have recently received considerable attention. In particular, stochastic resonance (SR) \([1, 2]\) has been studied in various systems. The resonance response of a noisy nonlinear system to a subthreshold signal can be optimized by noise intensity. Several studies have reported SR-like behavior also in chaotic systems, both numerically and experimentally \([3, 4, 5, 6, 7]\). These resonance behaviors are called deterministic SR (DSR), and it is often interpreted that chaos intrinsically generates the noise in the SR scenario. In some DSR studies, the researchers insisted that crises have an important relationship with SR. For example, DSR has been reported in a deterministic chaotic map that shows a two attractors merging crisis at a critical point \([3, 5, 6]\) and DSR is observed with synchronization of the attractor switching and an injected periodic signal. SR can be found in bistable systems not only of fixed points but of any kind of attractors, even chaos.

We have reported DSR in phase slips that occur when phase synchronization fails in a sinusoidally forced Rössler oscillator \([8, 9]\). When the forcing frequency is close to the natural frequency of a Rössler oscillator, the two phases are synchronized in the sense that the difference between these phases stays within a given region \([10, 11, 12]\). However, when the difference between their frequencies increases the phase synchronization breaks down and intermittent slips in the phase difference occur. These intermittent phase slips, where the phase difference quickly changes by 2\(\pi\), are considered to be jumps to another chaotic attractor in the phase space. The transition between phase synchronization and phase slips is caused by an unstable-unstable pair bifurcation crisis \([13]\). The DSR mechanism can be explained in terms of the synchronization between the chaotic attractor switching and the modulation of a bifurcation parameter by an injected periodic signal.

Here we demonstrate enhanced DSR resulting from coupling between forced Rössler oscillators. Similar to previous studies on coupled resonators \([14, 15, 16, 17]\), coupling synchronizes resonator behavior and the resonance response is enhanced as a result. This paper is organized as follows. In the next section, we introduce the coupled forced Rössler oscillators. In Sect. 3 we show the enhancement of DSR when coupling is introduced. Section 4 contains concluding remarks.

2. Model system

The DSR using phase slips has been studied in Ref. \([8, 9]\) in detail. Here we focus on the following coupled resonators to study the enhanced DSR achieved by coupling.

\[
\begin{align*}
\dot{x}_i & = s_i(-\nu_i y_i - z_i), \\
\dot{y}_i & = s_i(\mu x_i + a y_i + K[1 + \epsilon \sin(\omega t)] \sin(\Omega t) + \sum_{j=1}^{N} \frac{C}{N} (y_j - y_i)), \\
\dot{z}_i & = s_i(b + z_i(x_i - c)).
\end{align*}
\]  

(1)

Here \(s_i = 1 + \alpha(r_i^2 - \bar{r}^2)\), where \(r_i = \sqrt{x_i^2 + y_i^2}\), and \(\bar{r}\) and \(\alpha\) are constants whose meaning is explained later.
We set the system parameters $a = 0.2$, $b = 0.2$, and $c = 4.8$. The periodic forcing term $K \sin(\Omega t)$ is fixed as $K = 0.07$ and $\Omega = 1.077$ throughout this paper. The signal term $\epsilon \sin(\omega t)$ is also fixed as $\epsilon = 0.05$ and $\omega = 6.0 \times 10^{-4}$. We define the frequency average $\nu_m$ and the difference $\Delta \nu$ among the natural frequencies $\nu_i$, $i = 1, \ldots, N$ of each oscillator as $\nu_m = \frac{1}{N} \sum_{i=1}^{N} \nu_i$ and $\Delta \nu = \frac{1}{N} \sum_{i=1}^{N} |\nu_i - \nu_m|$. In this paper, the frequency average $\nu_m$ and the coupling strength $C$ are used as control parameters.

Here we focus on the phases of the chaotic oscillators. The projection of the Rössler attractor on the $x$-$y$ plane forms a ring around the origin. To describe the phase $\theta_i$ of the chaotic oscillator, we define $\theta_i = \arctan \frac{y_i}{x_i}$. We then define the phase difference between the Rössler oscillator and the forcing as $\phi_i = \theta_i - \Omega t$.

Although it is not so significant in this paper, we mention the meaning of factor $s_i$ [8, 9]. The factor $s_i$ enlarges the variation of the angular velocity, driven by the variation of the amplitude. Therefore, as $\alpha$ increases, the variation of the angular velocity increases. Phase slips occur more frequently as $\alpha$ increases. In the following, we set $\alpha = 0.002$. $\tau$ is the average $r$ value for an ordinary Rössler oscillator ($N = 1$, $\alpha = K = C = 0$, $\nu = 1$).

For the parameter regime studied in this paper, the oscillator phase $\theta_i$ intermittently breaks synchronization with the phase $\Omega t$ of the periodic forcing and the phase difference occasionally jumps $2\pi$. We observe DSR in the behavior of this phase slip. We consider each forced Rössler oscillator as a unit resonator. In this section, we choose $N = 2$ for explanation as the simplest coupled resonator system. As in the single resonator case described in Ref. [8, 9], the difference between the forcing frequency $\Omega$ and the average frequency $\nu_m$ of the Rössler oscillator governs the average rate of phase slips. We consider the average rate of phase slips to be the strength of the internal fluctuations, or the stochastic “noise” in the DSR scenario. In this paper, we fix $\Omega$, and use $\nu_m$ as the parameter characterizing the strength of this “noise”.

To study phase slip synchronization, we focus on the resonator phase difference $\Delta \phi$ between $\phi_1$ and $\phi_2$, that is, $\Delta \phi = \phi_1 - \phi_2 = \theta_1 - \theta_2$. For now, we consider the case without the signal ($\epsilon = 0$) unless otherwise stated. If the coupling strength $C = 0$, the phase slips of each forced oscillator occur intermittently and independently of each other. In this situation, $\phi_i$ occasionally jumps $2\pi$ independently, while $\Delta \phi$ increases and decreases by $2\pi$ independently. Intermittent synchronization of phase slips also occurs for non-zero coupling strength $C$ when the coupling strength $C$ is too small, as in Fig. 1(a) and (c). When the coupling strength $C$ exceeds a critical value, the phase slips of each forced oscillator occur intermittently but the slips of both resonators are synchronized with each other. Although phase slips occur with respect to the forcing, the resonator phase difference $\Delta \phi$ is always within a certain range, as in Fig. 1(b) and (d) [18].

3. Coupling enhancement of DSR

This section provides numerical results on DSR. We consider $N = 2$, unless otherwise stated. The interslip interval distributions (ISIDs) $\rho_i$ ($i = 1, 2$) for each resonator are shown in Fig. 2. The frequency difference $\Delta \nu$ results in different forms of ISIDs. When the forcing is modulated by an external periodic signal with frequency $\omega$, the distribution appears modulated with peaks at integer multiples $\tau_n$ ($= 2\pi n / \omega$, $n = 1, 2, \ldots$) of the modulating signal period. SR is observed as change in $\rho_i$ [19]. To quantify the strength of the resonance response, we use the difference in the ISID probability at $\tau_1$, with and without a signal, i.e. $\Delta \rho_i = \rho_{i, \epsilon > 0}(\tau_1) - \rho_{i, \epsilon = 0}(\tau_1)$.

In the case of coupled resonators, the behavior of the ISIDs, the exponential decay without a signal and the exponential decay of the multipeak envelope with a signal, are similar to the single resonator result. The difference between the distributions become closer
when the coupling strength increases, both with and without a signal ($\epsilon = 0$ and $> 0$). When the slips are perfectly synchronized with each other with sufficient coupling strength, the ISIDs for each resonator overlap and are coincident.

Figure 3 (a) shows the resonance strength $\Delta \rho$ versus $\nu_m$ when $\Delta \nu = 0$. $\Delta \rho$ has a maximum value at a particular value of $\nu_m$, which represents the strength of the internal fluctuation. The 1st resonator is shown by lines. The 2nd resonator shown by dotted lines. Figure (a) shows $\Delta \rho$ when $\Delta \nu = 0$ and Fig. (b) shows $\Delta \rho$ when $\Delta \nu = 0.0001(> 0)$.

Figure 4: Resonance strength $\Delta \rho_1$ as a function of coupling strength $C$ at a fixed $\nu_m$. It can be seen that there is an optimal coupling strength $C_{\text{opt}}$ which produces the maximal resonance response in the $\Delta \rho_1$ value. This optimality is qualitatively discussed later in Sect. 4. Figure 5 shows that similar coupling-enhanced response occurs for a larger number of oscillators, $N = 16$, when $\Delta \nu = 0$. There is an optimal $\nu_m$ in Fig 5 (a) and there is an optimal coupling strength $C_{\text{opt}}$ in Fig 5 (b), which produces the maximal resonance response.
4. Conclusion

We have shown numerically that coupling can enhance the deterministic stochastic resonance (DSR) response. In this sense, our result on enhanced resonance is consistent with the results of other studies of coupling-enhanced SR [14, 15, 16, 17]. However, to our knowledge, this is the first report of enhanced DSR by coupling in a deterministic system.

This enhancement can be described qualitatively as follows. When an external signal modulates the forcing in this system, the DSR effect is seen as statistical synchronization between the intermittent phase slips for each resonator and the external signal, manifest as resonance peaks in the interslip interval distribution (ISID). When the coupling is larger than a critical value, phase slips between resonators are synchronized with each other, causing the ISIDs of the intermittent phase slips in the resonators to be similar even when the oscillator frequencies are different, and the DSR effect is enhanced.

Increasing of coupling strength increases the average rate of slips, that is the internal fluctuation. In the case of standard SR, the increasing of coupling strength only suppresses effective noise intensity. In this sense, there is a difference in the effect of the coupling strength on fluctuations in the standard SR and DSR in the coupled forced Rössler oscillators. Detail behavior about effects of coupling will be reported elsewhere.

References