Transient Dynamics Observed in Strongly Nonlinear Mutually-Coupled Oscillators

Yuhki Aruga and Tetsuro Endo

Department of Electronics and Communication, Meiji University
Higashi-Mita, Tama, Kawasaki, Japan
Phone:+81-44-934-7344, Fax:+81-44-934-7344
Email: aruga@chaos.mind.meiji.ac.jp, endoh@isc.meiji.ac.jp

Abstract— In this paper, the behavior of the double-mode oscillation in mutually coupled oscillators with hard nonlinearity with the increase of ε (=degree of nonlinearity) is investigated. Namely, when ε becomes large, the two frequency components of the double-mode oscillation get close, and eventually they becomes a periodic solution via synchronization. In particular, just before the double-mode oscillation pulled in a periodic solution, we observed an interesting phenomenon such as two symmetric periodic solutions switch alternately. We elucidate its mechanism by calculating unstable manifolds of degenerate saddles.

1. Introduction

It is well-known that mutually coupled oscillators with hard nonlinearity can possess three attractors for small ε; namely, the same-phase, the reverse-phase and the double-mode attractors [1]. The same-phase attractor is a periodic solution in which two oscillators are synchronized with the same-phase, and the reverse-phase attractor is a periodic solution in which two oscillators are synchronized with the reverse-phase. The double-mode attractor is a quasi-periodic oscillation whose component frequencies consist of the same-phase frequency and the reverse-phase frequency. When a parameter ε, which presents the degree of nonlinearity, is increased, the same-phase solution becomes unstable via the pitchfork bifurcation for a certain value of $\varepsilon = \varepsilon_c$, while the reverse-phase solution is kept stable for all $0 < \varepsilon < 1$. The two frequency components of the double-mode oscillation get close as ε is increased, and eventually it becomes a periodic solution via synchronization which exists only for large ε [2]. In the last part of ε for which the double-mode solution exists (or just before pulled in the periodic solution), an interesting phenomenon such as two symmetric periodic solutions switch alternately, is observed. This is a "Transient Dynamics" widely noticed as typical nonlinear non-equilibrium phenomena in self-reproduction process. The transient dynamics reported so far is considered as quasi-stable states and observed on its way to an eventual steady state. However, in our example, there is no eventual steady state, and the system repeats the quasi-steady states permanently. To our knowledge, such an example of transient dynamics has not been reported. In this paper, it is elucidated that such dynamics is due to a heteroclinic cycle connecting degenerate saddles, which appear at the edges of aligned saddle-node bifurcations [3].

2. Switching Phenomena of Two Periodic Solutions

Two inductively-coupled oscillators with hard nonlinearity can be written by the following 4th-order autonomous system:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -\varepsilon(1 - \beta x_1^2 + x_1^4)x_2 - x_1 + \alpha x_3 \\ \dot{x_3} = x_4 \\ \dot{x_4} = -\varepsilon(1 - \beta x_3^2 + x_3^4)x_4 - x_3 + \alpha x_1 \end{cases} \tag{1}$$

where x_1 denotes the normalized output voltage of one oscillator, x_2 is its derivative, and where x_3 denotes the normalized output voltage of the other oscillator, x_4 is its derivative. The parameter $0 < \alpha < 1$ is a coupling factor; namely, $\alpha = 1$ means maximum coupling, and $\alpha = 0$ means no coupling. The parameter β controls amplitude. Via averaging method, the following results are obtained: for $\beta < 2\sqrt{2}$ the only stable state is no oscillation; for $\beta > 2\sqrt{2}$ the same-phase, the reverse-phase and no oscillation solutions are stable; for $4\sqrt{5}/3 < \beta < 4$, besides the same-phase, the reverse-phase, and no oscillation solutions, the double-mode solution (2-torus) consisting of the same-phase and the reverse-phase frequency components, becomes stable.

This double-mode solution presents various phenomena, depending on the coupling factor α as ε becomes large. From averaging method, the angular frequency of the same-phase solution is predicted as $\omega_1 = \sqrt{1-\alpha}$ and that of the reverse-phase solution is predicted as $\omega_2 = \sqrt{1+\alpha}$. Therefore, for α nearly equal to zero, ω_1

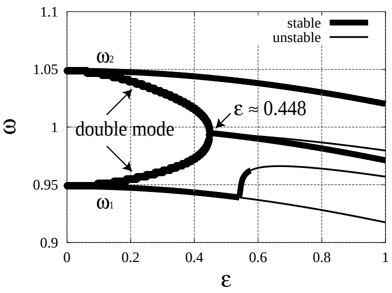


Figure 1: The variation of ω in terms of ε for $\alpha = 0.1$ and $\beta = 3.1$.

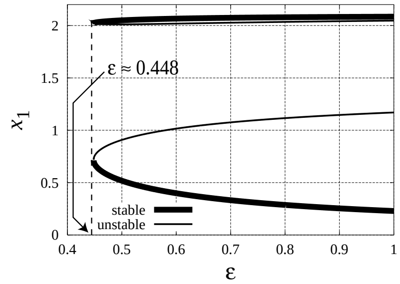
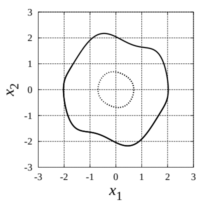
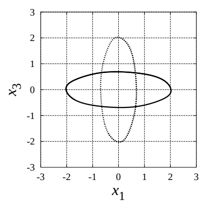


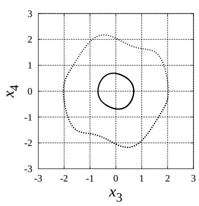
Figure 3: The disappearance of the synchronized solution via saddle-node bifurcation for $\alpha = 0.1$ and $\beta = 3.1$.



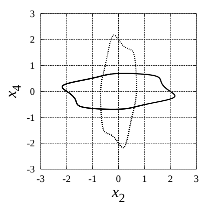
(a) The (x_1, x_2)-plane.



(b) The (x_1, x_3)-plane.

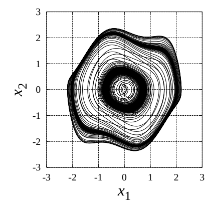


(c) The (x_3, x_4)-plane.

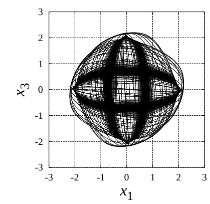


(d) The (x_2, x_4)-plane.

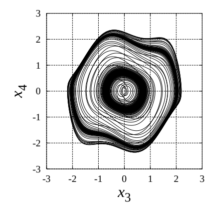
Figure 2: Two symmetric synchronized solutions for $\alpha = 0.1$, $\beta = 3.1$ and $\varepsilon = 0.449$.



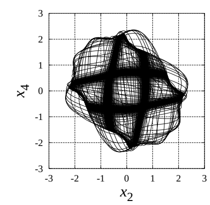
(a) The (x_1, x_2)-plane.



(b) The (x_1, x_3)-plane.



(c) The (x_3, x_4)-plane.

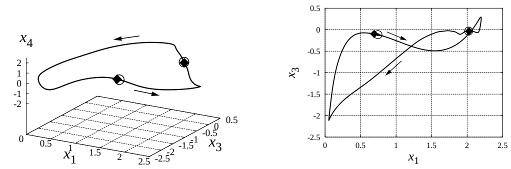


(d) The (x_2, x_4)-plane.

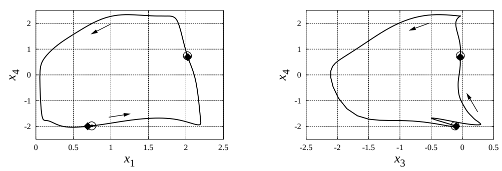
Figure 4: The switching phenomenon of the double-mode oscillation for $\alpha = 0.1$, $\beta = 3.1$ and $\varepsilon = 0.447$.

and ω_2 are close. As an example, we present the variation of ω_1 and ω_2 in terms of ε for $\alpha = 0.1$ and $\beta = 3.1$ in Fig. 1. In this case, as ε becomes larger, the two angular frequency components starting from the predicted value from averaging method, get closer and eventually around $\varepsilon \approx 0.448$ they coalesce to be one frequency. Namely, for $\varepsilon > 0.448$, there exists no double-mode oscillation, and instead there exists a single mode oscillation in which two frequency components of the double-mode oscillation are synchronized. We call this mode of oscillation a synchronized mode. From physical point of view, one oscillator is in oscillating state and the other oscillator is in no-oscillating state in this synchronized mode. Therefore, there is a large limit cycle in the (x_1, x_2)-plane and a small limit cycle in the (x_3, x_4)-plane and vice versa as seen in Fig. 2. Theoretically, the synchronized mode has two states from the symmetric property of (1); namely, (1) is invariant by replacing the variable x_1 by x_3 and x_2 by x_4. Namely, there are two solutions represented by solid curve and by dashed curve in Fig. 2, which appear from the symmetry of (1). Fig. 3 presents a bifurcation diagram of two symmetric solutions in terms of ε. From Fig. 3 it is noticed that two symmetric solutions disappear at the same value of $\varepsilon \approx 0.448$ via saddle-node bifurcation. Just after the disappearance of two solutions via saddle-node bifurcation ($\varepsilon < 0.448$, $\varepsilon \approx 0.448$), namely, in the last portion of the double-mode oscillation, a switching phenomenon between the two symmetric solutions in Fig. 2 occurs as in Fig. 4. In this paper we investigate the reason for this phenomenon.

Taking a Poincare section at $x_2 = 0$ in (1), flows in 4 dimensional phase space become discrete maps in 3 dimensional phase space (x_1, x_3, x_4). By using this discrete dynamical system, the unstable manifold W_u of the saddle about the saddle-node bifurcation point can be calculated by computer simulation as shown in Fig.



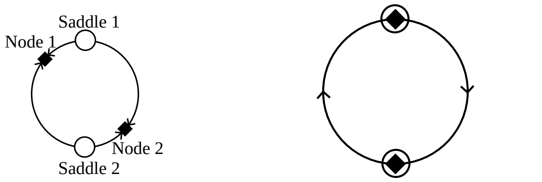
(a) The (x_1, x_3, x_4)-space. (b) Projection onto the (x_1, x_3)-plane.



(c) Projection onto the (x_1, x_4)-plane. (d) Projection onto the (x_3, x_4)-plane.

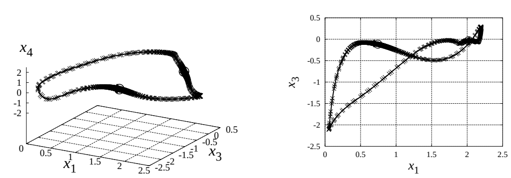
node: ♦, saddle: ⊙

Figure 5: The unstable manifold near the saddle-node bifurcation point for $\alpha = 0.1$, $\beta = 3.1$ and $\varepsilon = 0.449$.

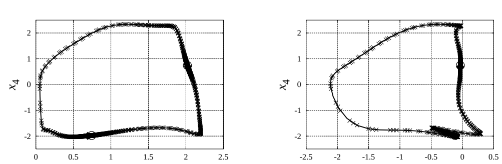


(a) After a saddle-node bifurcation point. (b) At a saddle-node bifurcation point.

Figure 6: The schematic diagram of a heteroclinic cycle.



(a) The (x_1, x_3, x_4)-space. (b) Projection onto the (x_1, x_3)-plane.



(c) Projection onto the (x_1, x_4)-plane. (d) Projection onto the (x_3, x_4)-plane.

node: ♦, saddle: ⊙, Poincare map: ×

Figure 7: The unstable manifold for $\varepsilon = 0.449$ and the attractor on the Poincare map for $\varepsilon = 0.447$ ($\alpha = 0.1$, $\beta = 3.1$).

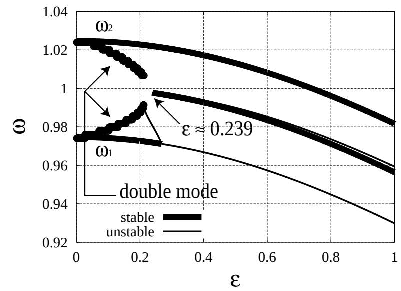
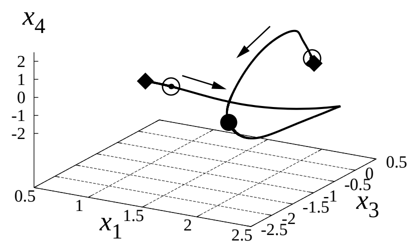


Figure 8: The variation of ω in terms of ε for $\alpha = 0.05$ and $\beta = 3.2$.



node: ♦, saddle: ⊙, reverse-phase solution: •

Figure 9: The unstable manifold near the saddle-node bifurcation point for $\alpha = 0.05$, $\beta = 3.2$ and $\varepsilon = 0.25$.

5, which can be explained by using a schematic diagram of Fig. 6. Near the saddle-node bifurcation point there is a general structure such that one W_u of Saddle 1 goes in Node 1, and the other W_u of it goes in Node 2, furthermore that one W_u of Saddle 2 goes in Node 1 and the other W_u of it goes in Node 2. At saddle-node bifurcation point, a node and a saddle coalesce to be a degenerate saddle, and therefore, there is a heteroclinic cycle connecting two degenerate saddles at this point. The structure such that a W_u of a saddle connects two stable Node 1 and Node 2 is almost kept due to continuity of vector field even after the disappearance of the synchronized solution via saddle-node bifurcation. In such a case, since nodes are already disappeared, nodes must be replaced by their traces. After all, the switching phenomenon shown in Fig. 4 can be explained as the following transient phenomenon: a flow stays traces of Node 1 and Node 2 for a long time and quickly moves along a heteroclinic cycle connecting two traces. To confirm that flows move certainly along the unstable manifold, a Poincare map of the attractor for $\varepsilon = 0.447$, which is the last part of the double-mode oscillation, and the heteroclinic cycle at $\varepsilon = 0.449$ are overlapped. Fig. 7 proves that flows move along the heteroclinic cycle.

Such a switching phenomenon does not always occur.

For example, for $\alpha = 0.05$ and $\beta = 3.2$ the double-mode oscillation in the ε versus ω plane cannot be connected to the synchronized solution as seen in Fig. 8. In such a case, when the synchronized solution is lost ($\varepsilon < 0.239$, $\varepsilon \approx 0.239$), the flow goes to the reverse-phase solution. This phenomenon can also be explained from the behavior of the unstable manifold of the saddles as follows. Fig. 9 presents the computer-generated W_u at $\varepsilon = 0.25$ about the saddle-node bifurcation point. It is noticed that one branch of W_u goes to a stable fixed point representing the reverse-phase solution. Therefore, after the disappearance of the synchronized solution via saddle-node bifurcation, the reverse-phase solution appears.

3. Switching Phenomenon for the Period-2 Solution

When ω_1 and ω_2 are separated considerably due to large value of α, and hence the 1 to 1-synchronization is impossible, the m versus n-synchronization occurs in general. In the case of $\alpha = 0.5$ and $\beta = 3.2$, the double-mode solution is synchronized about $\varepsilon \approx 0.881$ at the ratio of $\omega_1 : \omega_2 = 1 : 2$. In this case there coexist two symmetric synchronized solutions, and the double-mode solution just before pull-in, presents the switching between these two symmetric solutions like the case of $\alpha = 0.1$ and $\beta = 3.1$ [2]. For ε just before pull-in, the flow stays one of the period-2 solutions for a long time, and suddenly switches to the other one, and repeats this process permanently. Moreover the largest Lyapunov exponent of this flow is positive, therefore the switching attractor can be regarded as chaos. Furthermore, when ε goes beyond the bifurcation point, one of two sets of periodic solutions appears. Fig. 10 presents the ε versus x_1 bifurcation diagram in which curves A and A' correspond to one pair of the period-2 solution, and curves B and B' correspond to the other pair of it. It is noticed that they present a saddle-node bifurcation at the same value of $\varepsilon \approx 0.881$. Fig. 11 demonstrates four degenerate saddles and their unstable manifolds about the saddle-node bifurcation point. Note that the unstable manifolds certainly connect degenerate saddles. Namely, there is a heteroclinic cycle connecting each saddles such as $A \to B \to A' \to B' \to A$. The Poincare maps of the attractor are spread over this unstable manifold. From these results, it is elucidated that the transient dynamics occur along the heteroclinic cycle.

4. Conclusions

In this paper, the mechanism of transient dynamics observed in mutually-coupled oscillators with hard nonlinearity is elucidated. Namely, the key mechanisms of this phenomenon are: existence of symmetric periodic solutions aligned multiple saddle-node bifurcations, formation of the heteroclinic orbit connecting the degenerate saddles at the bifurcation point. Such transient dynamics can be seen in various nonlinear systems but its analysis has not been performed so far. In the near future, we will investigate transient dynamics such that multiple chaos sets are connected with unstable manifolds, which are deeply related to "Chaotic Itinerant".

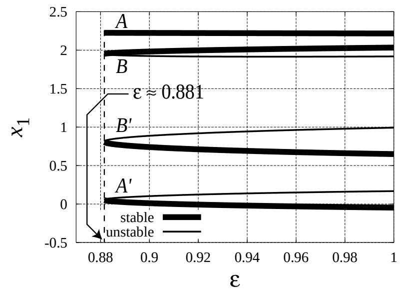


Figure 10: The disappearance of the synchronized solution via saddle-node bifurcation for $\alpha = 0.5$ and $\beta = 3.2$.

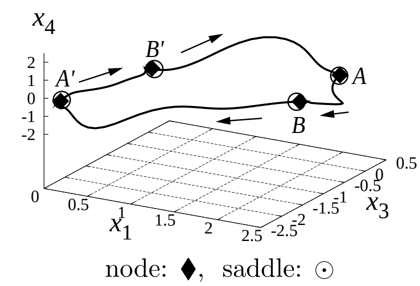


Figure 11: The unstable manifold near the saddle-node bifurcation point for $\alpha = 0.5$, $\beta = 3.2$ and $\varepsilon = 0.882$.

References

[1] T. Endo and T. Ohta, "Multimode Oscillations in a Coupled Oscillator System with Fifth-Power Nonlinear Characteristics," *IEEE Trans. Circuits and Systems*, Vol. CAS-27, No. 4, pp. 277–283, 1980.

[2] Y. Aruga, T. Endo and A. Hasegawa, "Instability of the Same-Phase Solution in a System of Two Coupled Oscillators," *Proc of NOLTA2001*, 2001.

[3] Y. Nishiura, "Nonlinear Problem 1," Iwanami, Chap. 6, 1999 (in Japanese).