Zero crossing statistics of chaotically jittered clock signals

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Abstract

In this paper we investigate the applicability of randomly or chaotically jittered clock signals to digital circuits. To do that, we characterize the random variable τ, representing the time interval between subsequent zero crossing points of the jittered clock, by determining its probability density and by computing its mean value and variance in some cases particularly interesting for what concerns the reduction of Electro Magnetic emission. Finally we will highlight some properties which depends on the statistical characterization of the modulating signal.

1. Introduction

Among the many applications of chaos in information engineering, improvement of EMC of certain periodically operated systems is recently attracting the attention of researchers.

This method consists in the injection of a chaos-based jitter so to avoid strict periodicity and thus high peaks in power spectral density.

Such a method has been proved effective in increasing Electro Magnetic Compatibility of switching signals employed, for example, in loudspeakers [2], [3] and in induction motor drivers [5], [1].

In such frameworks the invariance of the global system behavior with respect to the injected jitter can be often guaranteed.

On the contrary, if the same technique is applied to the jittering of clocks in digital (especially high-frequency) systems, several key facts must be taken into account. In particular (1)stricter specifications on the maximum tolerated deviation from the nominal frequency must be imposed; (2)impossibility of operating the system safely beyond a certain speed must be considered; (3)impact of possible frequency reduction on the overall processing speed of the system must be evaluated. We here address the second and third point, investigating the case in which the clock is frequency modulated by a PAM signal carrying samples both from system generating independent symbols and from a chaotic trajectory.

It actually turns out that the performance depends on the statistics of the zero-crossing instants of the jittered clock signals

With this, and by taking care on the nonlinear relationship between frequency modulation and time-domain characterization of signals, we analyze both systems in which the frequency deviation is constrained to assume only two values and systems with a instantaneous frequency that can be continuously distributed in a certain range.

Numerical simulation will be presented and compared with theoretical results highlighting differences between the possible approaches.

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2. Zero crosses distribution in frequency modulation

In this section, we investigate the distribution of the time between two successive zero crossings in the frequency modulated signal

$$S(t) = \cos\left[2\pi\left(f_0 t + \Delta f \int_{-\infty}^{t} \xi(\tau) d\tau\right)\right] \tag{1}$$

which represents the fundamental harmonic of a digital clock, so that zero crossing instants will correspond to the clock edges. Moreover Δf indicates the frequency deviation and the modulating signal is $\xi(t) = \sum_{k=-\infty}^{\infty} x_k g(t-kT)$ where $g(t)$ is a unit rectangular pulse $g(t)$ of duration T. As far as the symbols x_i are concerned, we consider two different cases: $(1)$the symbols x_k are i.i.d and distributed according to a certain density $\bar{\rho}_x : [-1,1] \mapsto \mathbb{R}^+$; $(2)$the symbols x_k are deterministic and generated by the evolution of a chaotic map, so that $x_{i+1} = M(x_i)$. They are distributed according to the invariant probability density of the map M [4], which we also indicate with $\bar{\rho}_x$.

Let us rewrite the band-pass phase-continuous signal $S(t)$ in a more useful form as

$$\begin{aligned} S(t) &= \\ &= \sum_{k=-\infty}^{\infty} \cos\left[2\pi\left(f_0 t + \Delta f x_k (t-kT) + \varphi_k\right)\right] g(t-kT) \end{aligned} \tag{2}$$

where the quantities $\varphi_k = 2\pi T \Delta f \sum_{j=-\infty}^{k-1} x_j$ are defined to guarantee the fulfillment of the phase-continuity constraint. Further more, under certain conditions (i.e T is not an integer multiple of all the instantaneous frequency), it can be shown that φ_k is a random variable uniformly distributed in $[0, 2\pi]$. Clearly, zero crossings occur each time the phase of the previously defined signal becomes an integer multiple of π. From now on let us indicate with τ the duration of the time interval between two subsequent zero crossings. In these conditions, the probability density of τ may be written as

$$P(\tau) = \Pr(b)P(\tau \mid b) + \Pr(n_b)P(\tau \mid n_b) \tag{3}$$

where $\Pr(b)$ is the probability that the time interval between two successive zero crosses contains a frequency change, and $P(\tau \mid b)$ is the corresponding conditioned probability density. $\Pr(n_b)$ and $P(\tau \mid n_b)$ have the analogous meaning for time intervals that lay on the same frequency. Obviously, the normalization condition of probability measures says that $\Pr(b) + \Pr(n_b) = 1$. Moreover, we will only consider $T > \frac{1}{f_0}$.

2.1. Determination of $\Pr(b)$ and $\Pr(n_b)$

Now we will determine the general expression of $\Pr(b)$, which depends on the statistic properties of the modulating symbols. We consider that, for a fixed half period $A = \frac{1}{2(f_0+x_i\Delta f)}$, corresponding to the frequency $f = f_0 + x_i\Delta f$, the mean number of intervals between two successive zero crossings inside T is $\frac{T}{A}$. Only one of them contains a frequency change. Recalling the relationship between the symbols x_i and the period A, that is $x_i = \frac{1-2Af_0}{2A\Delta f}$, and defining $\rho_A(A)$ as the probability distributions of the half periods A we have

$$\rho_A(A) = \frac{\bar{\rho}_x(x)}{|\frac{dA}{dx}|} = \frac{1}{2A^2\Delta f}\bar{\rho}_x(\frac{1-2Af_0}{2A\Delta f}) \quad (4)$$

where $\bar{\rho}_x(x)$ is the probability distribution of the symbols x_i. Finally, weighting with the symbols probability density and integrating over all the possible periods we obtain

$$\Pr(b) = \left[\int_m^M \frac{T}{A}\rho_A(A)dA\right]^{-1} \quad (5)$$

where $m = \frac{1}{2(f_0+\Delta f)}$ is the duration of the smallest interval between two successive zero crossings, that is the half period which corresponds to the maximum frequency and viceversa $M = \frac{1}{2(f_0-\Delta f)}$. Substituting (4) in (5), we obtain

$$\Pr(b) = \left[\int_m^M \frac{T}{2A^3\Delta f}\bar{\rho}_x(\frac{1-2Af_0}{2A\Delta f})dA\right]^{-1} \quad (6)$$

which indicates, as expected, that $\Pr(b)$ increases as the period T of the modulating signal decreases. The normalization condition says that $\Pr(n_b) = 1 - \Pr(b)$.

2.2. Determination of $P(\tau \mid b)$ and $P(\tau \mid n_b)$

From the very definition of probability distribution we have that

$$\frac{\int_m^\theta (\frac{T}{\tau}-1)\rho_A(\tau)d\tau}{\int_m^M (\frac{T}{\tau}-1)\rho_A(\tau)d\tau} \quad (7)$$

indicates the probability distribution of τ, and where $\frac{T}{\tau} - 1$ is the mean number of time intervals of duration τ inside T, that excludes the partial half period immediately after the last frequency change as well as the one immediately before the next frequency change. In order to determine $P(\tau \mid n_b)$ we have to consider the derivative of (7) with respect to the variable θ. Then we obtain the density

$$P(\tau \mid n_b) = \frac{(\frac{T}{\tau}-1)\rho_A(\tau)}{\int_m^M (\frac{T}{\tau}-1)\rho_A(\tau)d\tau} \quad (8)$$

A different computation has to be done for the determination of $P(\tau \mid b)$ In fact, we must take into account the contribution of t_1, which is the time interval between the last zero cross and the frequency change, and t_2, which is the time interval between the frequency change and the following zero cross instant. Figure 1 explains the meaning of t_1 and t_2. The dotted line represents the instant when the frequency change occurs. The random variable $\tau \mid b$ can be written as

$$\tau \mid b = t_1 + t_2 = \phi A + (1-\phi)B \quad (9)$$

where $A = \frac{1}{2(f_0+x_i\Delta f)}$, $B = \frac{1}{2(f_0+x_{i+1}\Delta f)}$ are random variables with a certain distribution which depends on the statistic of the modulating symbols x_i, and ϕ is a random variable uniformly distributed in $[0, 1]$, that indicates the ratio between the time spent at the frequency $f_0 + \Delta f x_i$ and τ. Now we have to determine the probability density of $\tau \mid b$ once that the densities of the random variables that constitute it are known. We have to make a distinction based on assumptions we can make on x_i.

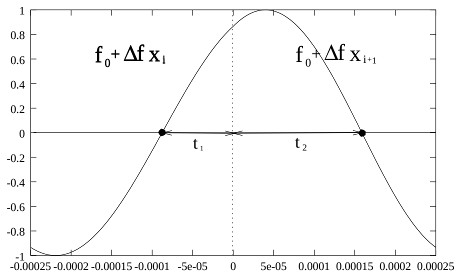

Figure 1: Meaning of t_1 and t_2. $f_0 = 2000$Hz, $\Delta f = 100$Hz, $x_i = 1$ and $x_{i+1} = -1$.

Let us consider first that x_i a sequence of i.i.d., uniformly distributed in $[-1, 1]$. Hence, with straightforward but long manipulations that take into account all possible values of ϕ, we derive

$$P(\tau \mid b) = 2\int_m^\tau \int_\tau^M \frac{\rho_A(A)\rho_B(B)}{B-A}dBdA \quad (10)$$

where the multiplication by 2 is due to the fact that A and B are independent, identically distributed random variables, and so $\Pr\{A > B\} = \Pr\{B > A\} = \frac{1}{2}$. In particular $\rho_A(y) = \rho_B(y) = \frac{1}{2y^2\Delta f}\bar{\rho}_x(\frac{1-2yf_0}{2y\Delta f})$.

On the contrary if the x_i are generated by the evolution of a chaotic map, the next symbol x_{i+1} is univocally determined by the knowledge of the previous one. So B is not an independent variable, but is a deterministic function of A. With this we may rewrite the expression (10) for chaotically dependent symbols as

$$P(\tau \mid b) = \int_{D_1} \frac{\rho_A(A)}{B(A)-A}dA + \int_{D_2} \frac{\rho_A(A)}{A-B(A)}dA \quad (11)$$

where the set is defined as $D_1 = \{A \mid A \in [m,\tau] \cap (B(A) \geq \tau)\}$, $D_2 = \{A \mid A \in [\tau, M] \cap (B(A) \leq \tau)\}$, and where $\rho_A(A) = \frac{1}{2A^2\Delta f}\bar{\rho}_x(\frac{1-2Af_0}{2A\Delta f})$. Note that here $\bar{\rho}_x$ is the invariant density of the map M so that the integral (11) has been explicitly splitted into the sum of two integrals clarifying the dependency of the symbols.

3. Application of the formula in two interesting cases

There are two particular symbols distribution $\bar{\rho}_x$ that, in conjunction with a proper period T of the modulating signal, appear to be very efficient in flattening spectral peaks, and thus suitable for the improvement of EMC. They are: (1) uniform distribution, that is $\bar{\rho}_x(x) = \frac{1}{2}$ if $x \in [-1, 1]$ and 0 elsewhere; (2) binary distribution, that is $\bar{\rho}_x(x) = \frac{1}{2}\delta(x-1) + \frac{1}{2}\delta(x+1)$, where $\delta(\cdot)$ is the Dirac Delta operator. In order to evaluate the loss of performance of a digital system it is convenient to define the average interval $\overline{\tau}$ between two zero cross as

$$\overline{\tau} = \Pr(b)\int_m^M \tau P(\tau \mid b)d\tau + \Pr(n_b)\int_m^M \tau P(\tau \mid n_b)d\tau \quad (12)$$

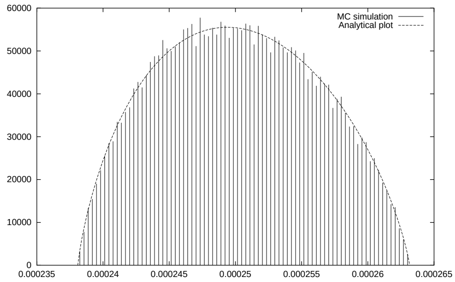

Figure 2: zero crossing distribution between a frequency change with independent uniformly distributed symbols. $f_0 = 2000\text{Hz}$, $m = 2.381 \times 10^{-4}s$, $M = 2.632 \times 10^{-4}s$

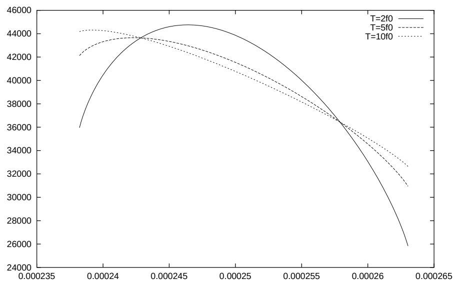

Figure 3: zero crossing distribution with independent uniformly distributed symbols. $f_0 = 2000\text{Hz}$, $m = 2.381 \times 10^{-4}s$, $M = 2.632 \times 10^{-4}s$

and the correspondent variance σ_τ^2 of τ, that is

$$\sigma_\tau^2 = \Pr(b)\int_m^M (\tau - \overline{\tau})^2 P(\tau \mid b)d\tau + \Pr(n_b)\int_m^M (\tau - \overline{\tau})^2 P(\tau \mid n_b)d\tau \tag{13}$$

We will deal separately with these two quantities, and we will further distinguish the case of independent and chaotic symbols.

3.1. Uniform distribution

3.1.1. Independent symbols

Since $\Delta f = \frac{M-m}{4Mm}$ and considering the general formulas obtained in the previous section to the uniform distribution case for when the symbols x_i are independent, we have

$$P_{\text{un}}(b) = \frac{8m^2M^2\Delta f}{T(M^2 - m^2)} = \frac{1}{2f_0T} \tag{14}$$

while for the probability densities we obtain

$$P_{\text{un}}(\tau \mid n_b) = \frac{2m^2M^2(\tau - T)}{\tau^3(M-m)[2mM - T(M+m)]} \tag{15}$$

$$P_{\text{un}}(\tau \mid b) = \frac{2}{3Mm\tau^3(M-m)^2}\big[Mm(M-m)\times \times(M-\tau)(m-\tau)\tau - m^3\tau^3\ln(\frac{(M-m)\tau}{(M-\tau)m}) - M^3m^3\ln(\frac{(M-\tau)m}{(\tau-m)M}) + M^3\tau^3\ln(\frac{(M-m)\tau}{(\tau-m)M})\big] \tag{16}$$

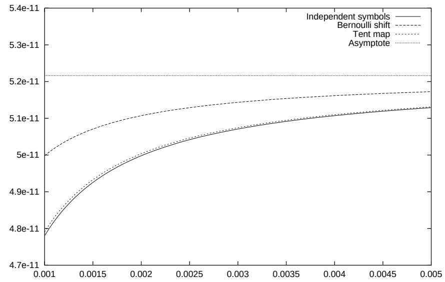

Figure 4: Variance of τ in different cases. $f_0 = 2000\text{Hz}$, $m = 2.381 \times 10^{-4}s$, $M = 2.632 \times 10^{-4}s$

Substituting (14),(15), (16) into equation (3) we have the total probability density of τ. Let us first consider the plot of $P_{\text{un}}(\tau \mid b)$, which is shown in figure 2. The precision of $P_{\text{un}}(\tau \mid b)$ as given by 16 has been tested by mean of extensive Montecarlo (MC) simulation. We note the presence of an absolute maximum near $\frac{1}{2f_0}$ and a decreasing trend going both towards m and M. The plots of $P(\tau)$ are shown in figure 3. We note that the concavity increases as T decreases. That is because $\Pr(b)$ is a decreasing function of T, and so the more T increases the less $P(\tau \mid b)$ gives its contribution to the $P(\tau)$ density.

In this case we have that $\overline{\tau}$ is a constant. Further calculations show that, for uniform distribution of the symbols we have $\overline{\tau} = \frac{1}{2f_0}$, independently of T. Instead, σ_τ^2 is an increasing function of T with an horizontal asymptote at the value $\frac{M^2+m^2-8Mmf_0(M-m)+8M^2m^2f_0^2\ln(\frac{M}{m})}{4(M-m)(M+m)f_0^2}$. The trend of the variance σ_τ^2 as a function of T is shown in figure 4.

3.1.2. Chaotically generated symbols

The symbols are supposed to be generated by a chaotic map, and uniformly distributed according to its invariant density $\bar{\rho}_x$. The unique expression which is sensitive to the use of dependent x_i is $P(\tau \mid b)$. In fact, while $\Pr(b)$, $\Pr(n_b)$ and $P(\tau \mid n_b)$ depend only on the first order statistic of the symbols, $P(\tau \mid b)$ depends on the second order statistic by considering the distribution of two successive symbols. As examples we show in figure 5 $P(\tau \mid b)$ when the symbols are generated by a Tent map, and in figure 6 $P(\tau \mid b)$ when the symbols are generated by a Bernoulli shift. The histogram is the result of a MC simulation with 50000 samples, while the continuous line correspond to the application of the equation (11). We note that there is a strong concentration of time intervals around the inverse of the frequency $\frac{1}{2(f_0+\frac{\Delta f}{3})}$, which corresponds to the fixed point with negative slope of the map. It is also worthwhile noting that while independent symbols have a $P(\tau \mid b)$ at the furthest points (as shown in figure 2), chaotically generated symbols have a a $P(\tau \mid b)$ which shows a clearly map dependent behaviour, and a finite value in correspondence with the fixed points of the map. Note also that $\overline{\tau}$, that could be expected to be map dependent, is actually independent of the second order statistics of the map. The variance σ_τ^2 is shown in figure 4 as a function of T with chaotic symbols generated both by a Tent map and by a Bernoulli shift. We note that, while the variance of τ with symbols generated by a Tent map is almost superimposed to the variance with independent symbols, the variance of τ with symbols generated by a Bernoulli shift is always greater.

Additionally, each curve converges, for $T \to \infty$, to the horizontal asymptote.

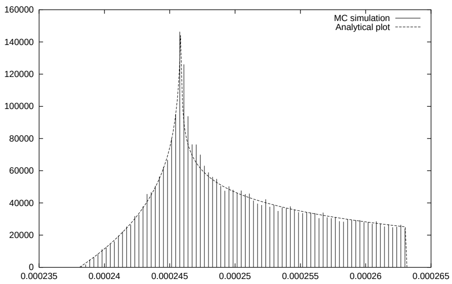

Figure 5: zero crossing distribution between a frequency change with symbols chaotically generated by a tent map. $f_0 = 2000$Hz, $m = 2.381 \times 10^{-4}s$, $M = 2.632 \times 10^{-4}s$

3.2. Binary distributed symbols

In this subsection we will not make distinction between binary independent symbols and binary chaotically generated symbols. In fact, it may be shown that the quantization of the symbols generated by a chaotic map with uniform invariant density $\bar{\rho}$ gives a sequence of independent symbols. Since we restrict our field the maps with uniform invariant density such a distinction is therefore useless.

The application the general formulas obtained in the previous section to the binary distribution leads to the following results

$$P_{\text{bin}}(b) = \frac{2}{T}\frac{Mm}{M+m} = \frac{1}{2f_0T} \tag{17}$$

while for the probability densities we obtain

$$P_{\text{bin}}(\tau \mid n_b) = \frac{(\frac{T-\tau}{\tau^3})(\delta(\tau-m)+\delta(\tau-M))}{\frac{T-M}{M^3}+\frac{T-m}{m^3}} \tag{18}$$

$$P_{\text{bin}}(\tau \mid b) = \frac{1}{4}(\delta(\tau-m)+\delta(\tau-M)) + \frac{1}{2(M-m)} \tag{19}$$

We note that $P(\tau \mid n_b)$ is zero everywhere, except for $\tau = m$ and $\tau = M$, while $P(\tau \mid b)$ have both impulsive contribution at the same instants, due to the succession of two equal symbols, and continuous uniform contribution due to the succession of two different symbols. Substituting the previous expressions (17),(18), (19) into equation (3) we have the total distribution of the time between two successive

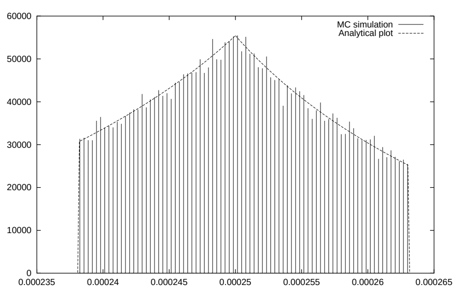

Figure 6: zero crossing distribution between a frequency change with symbols chaotically generated by a Bernoulli shift. $f_0 = 2000$Hz, $m = 2.381 \times 10^{-4}s$, $M = 2.632 \times 10^{-4}s$

zero crossing. The average value of τ is $\overline{\tau} = \frac{1}{2f_0}$, while σ_τ^2 is a always increasing function of T, which has an horizontal asymptote at the value $\frac{-2f_0^2Mm(M-m)+3T(M+m+4f_0Mm(-2+f_0M)+4f_0^2Mm^2)}{12f_0^2T(M+m)}$, as shown in figure 7. Finally, we note that the value of the horizontal asymptote with binary distributed symbols is always greater than the one with uniform distributed symbols.

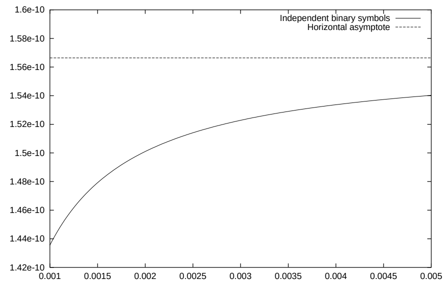

Figure 7: Variance of τ for binary independent symbols. $f_0 = 2000$Hz, $m = 2.381 \times 10^{-4}s$, $M = 2.632 \times 10^{-4}s$

4. Conclusions

In our investigation about statistic properties of the duration of the time τ between two edges of a frequency modulated clock, two results worth of note appeared: (1) The average value of τ depends on the first order statistic and is independent both of the second order statistic of the modulating PAM signal symbols and of its period T, (2) the variance of τ depends both on the second order statistic of the symbols and on the period T. In particular the variance of τ is an increasing function of T, with an horizontal asymptote at a value which depends on the first order statistic of the symbols. The rate of convergence to the asymptotic value, instead, depends on the second order statistic.

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