Estimation of Initial Conditions for Chaotic Systems

Emmanuelle Cosquer and Maciej J. Ogorzałek

University of Mining and Metallurgy
30-059 Kraków, Poland
Phone: +48-12-617-36-13, Fax:+48-12-634-48-25
Email: maciej@zet.agh.edu.pl

Abstract

In this paper we recall two methods for estimation of initial conditions for chaotic systems on the basis of measured time-series. We investigate in some detail the performance of these methods in the presence of noise for data generated by piecewise linear Markov maps. We propose further to use the Viterbi algorithm to improve the performance of these methods.

1. Introduction

An important problem in various application consists of processing data generated by non linear dynamical systems. In order to exploit chaotic signals, there is a need for algorithms which can give a good estimation of these signals especially in the presence of noise. In most cases, the estimation of chaotic signal is reduced to the knowledge of the initial conditions. Because chaotic signals are very sensitive to changes of initial conditions, these algorithms have to be as efficient as possible. The model we consider is as follows:

$$h(n+1) = f(h(n)) \tag{1}$$

Only signals scaled to the interval $[0, 1]$ will be considered, then a white Gaussian noise $w(n)$ will be added such that:

$$x(n) = h(n) + w(n) \tag{2}$$

Different methods have been proposed to estimate a chaotic signal embedded in noise, particularly in white Gaussian noise[1, 2, 7]. We will study only two methods : The Extended Halving Method [7] and the Symbolic Dynamics-Based method [1] in case of signals generated by Piecewise Linear Markov Maps. After describing the two methods, their performance will be evaluated in the presence of noise. Finally we will show how applying the Viterbi algorithm one could improve the estimation of the initial condition.

2. Mathematical Preliminaries

In order to understand the algorithms, we need to give some basic definitions.

- **Piecewise Linear Map**
  Let $I = [a, b]$ - a closed interval on the real line. A map $T : I \longrightarrow I$ is called Piece-Wise Linear if there exists a partition vector $p = (p_1, p_2, p_3, ...p_n, p_{n+1})$, $p_1 = a$ and $p_{n+1} = b$, such that for each $l \in \{1, 2, 3, ..., n\}$, T restricted to the segment (p_l, p_{l+1}) is a linear function. The set of partition points $\{p_1, p_2, ...p_n, p_{n+1}\}$ is denoted by Q.

- **"take partition points into partition points"**
  Let Q be the set of points of the partition vector p. We say [7] that the map $T : I \longrightarrow I$ takes partition points into partition points if $T(Q) \subset Q$. If T is discontinuous, we require $T(p_l^-)$ and $T(p_l^+)$ to be in Q. Here, $T(p_l^+)$ and $T(p_l^-)$ are the right and left limits of T at p_l.

- **Communication property**
  We say [7] that a partition vector p has the communication property under the map $T : I \longrightarrow I$ if for any subintervals $I_i = (p_i, p_{i+1})$, $I_j = (p_j, p_{j+1})$ there exist integers m_1 and m_2 such that $I_i \subset T^{m_1}(I_j)$ and $I_j \subset T^{m_2}(I_i)$

- **Piecewise Linear Markov Map** A map $T : I \longrightarrow I$ is called a piece-wise linear Markov map if it satisfies the following conditions for the fixed partition vector p.

  1. T is a piecewise linear map with respect to the partition vector p.
  2. $Inf_{x \in I} |\frac{dT}{dx}| > 1$,
  3. T takes partition points into partition points,
  4. The partition vector p has the communication property.

3. Algorithms for estimation of initial conditions

In the following considerations we will assume that the variance of the noise $w(n)$ is known.

3.1. Method based on Symbolic Dynamics

This method proposed originally by Cong *et al.* [1] is based on the symbolic sequence associated with the measured signal and on the backward iterations of the map. Let $E = \{E_0, E_1, \ldots, E_{q-1}\}$ be a finite disjoint partition of the phase space M. Assuming that at time n the point $h(n)$ is in the i^{th} element of the partition we assign to it a symbol $s(n) = i,\ i \in \{0, 1, \ldots, q-1\}$. According to this representation we can encode any orbit by a sequence $S = (s(0), s(1), \ldots, s(q) \ldots)$ The symbolic sequence determines the operation φ from M to the symbol space so, that

$$\varphi(h(0)) = S \iff h(0) \in f^{-n} E_{s(n)} \tag{3}$$

for $n \geq 0$, where $f^{-n} E_{s(n)}$ represents the set of points that are mapped to the set $E_{s(n)}$ after n iterations.

Let f_i - the restriction of f to E_i, and suppose that we know the symbolic sequence $\{s(0), s(1), \ldots\ldots, s(L-1)\}$ of the signal determined by the points $\{h(0), h(1), \ldots\ldots, h(L-1)\}$. We can limit the point $h(0)$ (the initial condition) to the set $\cap_{n=0}^{L-1} f^{-n} E_{s(n)}$. The larger value of L, the smaller will be this set.

The estimation of the initial condition will be given by the following equation :

$$\hat{h}(0|L) = f^{-1}_{s(0)} \circ f^{-1}_{s(1)} \circ \ldots \circ f^{-1}_{s(L-1)}(\xi) \tag{4}$$

where ξ is a point in the domain of $f^{-1}_{s(L-1)}$.

In the noisy case the symbolic sequence has to be estimated from the noisy signal. Thus the resulting sequence may be non-admissible, this means that it can happen that $f^{-1}_{s(n+1)} \circ \ldots \circ f^{-1}_{s(L-1)}(\xi)$ is not in the domain of $f^{-1}_{s(n)}$. In such a case, a modification which consists of taking the closest point which is admissible, is made. It is easy to notice that if the noise level is large it will be very hard to find the admissible sequence which corresponds exactly to the noise-free signal.

3.2. Extended halving method

This method is closely related to the decimal expansions of real numbers. The base-n expansions of real numbers can be formulated in terms of itineraries.

Let $E = \{E_0, E_1, \ldots, E_{q-1}\}$ be a finite disjoint partition of the phase space. r is the characteristic sequence defined as $r_i = 1$ if the slope of the map in E_i is greater then 0 and $r_i = -1$ if the slope of the map in E_i is less then 0. Let $h(0) \in [0, 1]$ – a given initial point. Let a_k be the itinerary of $h(0)$ under the iterations of f defined in such a way that $a^i_k = i + 1$ if the point $f^{k-1}(h(0)) \in E_i$.

Let $\{I_k\}_{k=0}^{\infty}$ be the sequence defined by $I_0 = 1$, $I_k = (I_{k-1} r_{a^i_k})$ then we can estimate the initial condition using

$$\hat{h}(0) = \sum_{k=1}^{\infty} \frac{c_k}{n^k} \tag{5}$$

where

$$c_k = \begin{cases} a_1 - 1, & \text{if } k = 1 \\ a_k - 1, & \text{if } I_{k-1} = 1 \text{ and } k \geq 2 \\ n - a_k, & \text{if } I_{k-1} = -1 \text{ and } k \geq 2 \end{cases} \tag{6}$$

In the noisy case, as in the previous method, we estimate all the sequences from the noisy signal and we use the same algorithm.

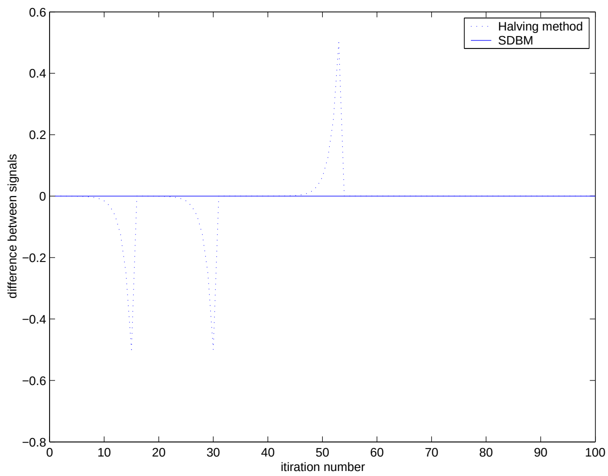

Figure 1: Comparison of the Symbolic Dynamics Based Method and the Halving method showing superior performance of SDBM.

3.3. Performance of the two methods

To evaluate the performance we reconstructed the signals by iterating the system equations starting from the initaial conditions estimated using the two abovementioned methods. We tried to find the error between the real signal and its reconstruction using the two methods in the case of very small noise level. Even in such a case the difference between the estimate of the initial condition and its real value is ca. 10^{16} for the halving method. It is almost 0 for the symbolic dynamic based method. Whatever the method or the map, in cases of high level of noise, the problem of estimation becomes more difficult. The estimation errors come from erroneous symbolic sequences. In the next part we will see how the Viterbi algorithm can improve the estimation.

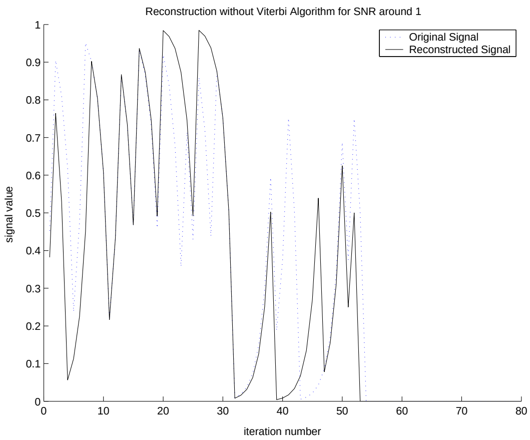

Figure 2: Reconstruction without Viterbi algorithm using the method of Cong *et al.*

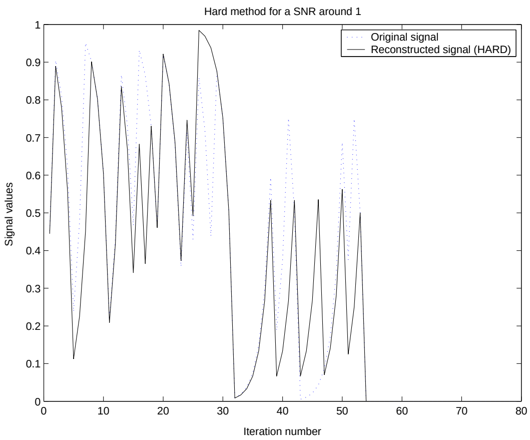

Figure 3: Reconstruction using the symbolic sequence method with Viterbi algorithm – hard version.

4. The Viterbi Algorithm

The Viterbi Algorithm was first proposed as a solution to the decoding of convolutional codes by Andrew J. Viterbi in 1967. This Algorithm can be interpreted as a dynamic progamming algorithm. It is used to minimize the error probability by comparing the likelihoods of a set of possible state transition that can occur, and deciding which of these has the highest probability of occurrence.

There are two variants of Viterbi algorithm [6]:

- the hard version, computes the most probable sequence, basing on the knowledge of the noisy signal and the noise variance by maximizing the following expression

$$p(s,y) = \prod_{k=1}^{N} p(s_k|s_{k-1})p(y_k|s_k) \quad (7)$$

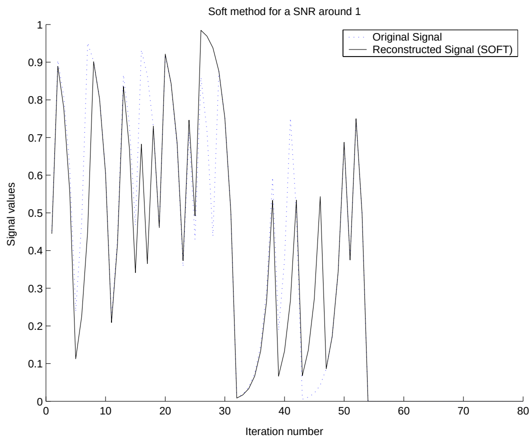

Figure 4: Reconstruction using the symbolic sequence method with Viterbi algorithm – soft version.

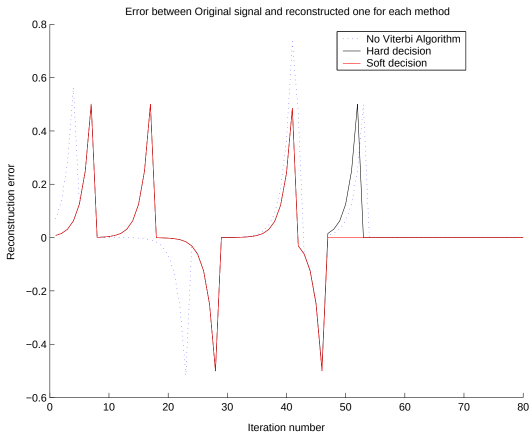

Figure 5: Error between the original signal and the signal reconstructed using soft and hard methods.

  where s is the symbolic sequence and y the noisy signal observed. s_k will take values according to a partition vector E as for the precedent algorithms, but for a better precision, the interval $[0,1]$ has to be divided into a higher number of subintervals.

- the soft version works directly on the points given by the noisy version of the signal. It computes the most probable sequence according to the map, without the knowledge of the noise.

The algorithm (hard and soft version) works in two main phases, a forward one and a backward one, and consists of finding the path which gives the highest probability.

As described previously the algorithms to estimate the initial condition are based on a symbolic/characteristic sequence. In the case of high level of noise, and sometimes even in case of low noise levels, this symbolic sequence determined from the measurement data are wrong, which results in bad estimation. The idea is to use the Viterbi algorithm to find a symbolic sequence as close as possible to the real one before using the symbolic dynamics-based method, in order to improve as much as possible the probability to have the correct symbolic sequence.

5. Experimental Results

We have carried out many experiments using the two versions of the Viterbi Algorithm. It is visible that the Viterbi algorithm in both versions improves the estimation of the initial condition.

The Fig.2 shows the results obtained without using the Viterbi algorithm. Results obtained using the Viterbi algorithm are shown in Figs 3 and 4 (in case of a noisy data-set generated by the Bernoulli shift map for a SNR -0.5dB). Fig.5 shows the comparison of performance.

The maximum error for each method was:

- With no Viterbi Algorithm : $E = 0.09737$
- With Hard descision : $E_h = 0.07373$
- With Soft descion : $E_s = 0.06161$

It is clear that the Viterbi Algorithm improves the results.

The Fig.6 shows the influence of SNR on the estimation of the initial conditions in the two cases: without the Viterbi algorithm and using its soft variant. In all cases we obtain better results using the Viterbi algorithm.

6. Conclusions

We have presented two methods for estimation of initial conditions of chaotic signals on the basis of measured time series. We have shown how to improve estimation of the initial condition for signals embedded in white Gaussian noise. Application of the Viterbi algorithm improves significantly the estimation but it should be underlined that the estimation is not always correct. Moreover this method involves the knowledge of the noise variance, which is not always easy to obtain.

Acknowledgments

This research has been supported by The EU Program COSYC of SENS HPRN-CT-2000-00158 and by the Polish State Committee for Research program SPUB-M (112/E-356/SPUB-M). The authors wish to thank Dr. Zbigniew Galias for his help in implementation of the Viterbi algorithm and useful comments which improved the manuscript.

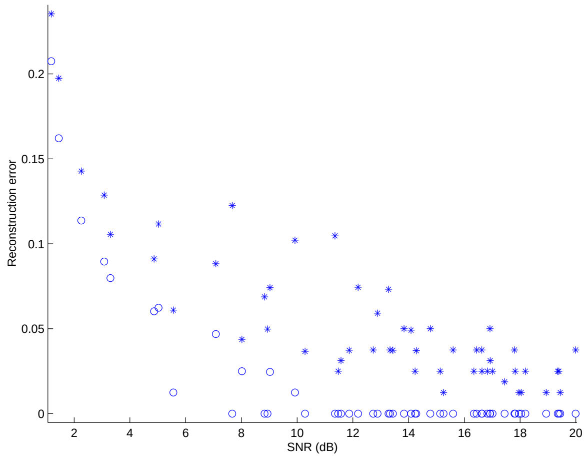

Figure 6: Influence of the SNR on the performance of the reconstruction algorithms. The circles indicate the results obtained using the soft version of Viterbi algorithm, the asterisks denote results obtained using original SDBM.

References

[1] L. Cong, X. Wu, S. Sun : A General Efficient Method for Chaotic Signal Estimation. IEEE Transaction on Signal Processing. Vol.47, No.5, 1999, pp.1424-1427.

[2] S. Kay, V. Nagesha : Methods for Chaotic Signal Estimation. IEEE Transaction on Signal Processing. Vol.43, No.8, 1995, pp. 2013-2016.

[3] M.P. Kennedy, R. Rovatti, G.Setti Eds.: Chaotic Electronics in Telecommunications. CRC Press, 2000.

[4] A. Lasota and M.C. Mackey: Chaos Fractals and Noise Stochastic Aspects of Dynamics. Springer-Verlag, 1994.

[5] R. Rovatti, G. Mazzini, G. Setti : A Tensor Approach to Higher Order Expectations of Quantized Chaotic Trajectories - Part I: General Theory and Specialization To Piecewise Affine Markov Systems. IEEE T-CAS- I, Vol. 47, No 11, 2000, pp. 1571-1583.

[6] M.S. Ryan , G.R. Nudd : The Viterbi algorithm. 12th February 1993.

[7] S. Wang , P.C. Wip, H. Leung : Estimating Initial Conditions of Noisy Chaotic Signals Generated by Piece-Wise Linear Markov Maps Using Itineraries. IEEE Transaction on Signal Processing, Vol.47, No.12, 1999, pp.3289 -3302.