Coded Modulations Using Chaotic Systems Controlled by Small Perturbations

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Abstract— Recently it has been observed that communications using chaos show best performance if the information production related to the chaos in the transmitter is controlled and used for the payload of the communication [5]. In the case of one-dimensional Markov maps, this can be achieved by controlling the chaotic system by means of small perturbations. In particular, the perturbations are applied in such a way that the generated (natural) symbolic sequence follows the information signal. The paper develops the link between communication schemes based on such chaotic systems and trellis coded modulations which are well understood in communication theory.

1. Introduction

In the present paper, the authors investigate communication systems deriving from one-dimensional Markov maps, in particular those based on maps that are topologically conjugate to the binary shift map (Bernoulli shift map). We show that in this case the resulting modulation scheme, involving a quantized dynamics, are equivalent to trellis coded modulations, introduced by Ungerboeck [6] in the 80ies, by now a standard approach in telecommunications for bandwith-optimized communications. Thus, the particular chaos based modulation system can be represented, as in the case of trellis coded modulations, by a convolutional code driving a signal constellation on the real line (PAM case) or on the complex plane (QAM and PSK case). Furthermore, the Viterbi algorithm can be applied in the receiver in order to recover the information signal at an acceptable computational complexity.

We show that for the shift map, there is an underlying trivial linear convolutional code, and we explore the relation of more powerful linear convolutional codes with equivalent maps. Furthermore, we investigate a special type of sub-optimal receiver of sliding window type, which can result in close-to-optimal performance of the receiver at a substantially reduced computational cost. Finally, we conclude by giving some new perspectives of coded modulation and channel coding techniques based on chaotic systems leveraging existing knowledge about structurally similar schemes in telecommunications.

2. Coded Modulations Based on Chaotic Maps

For the purpose of this paper, we will focus on a discrete time representation of digital modulations, which can be obtained through sampling from a number of relevant wire-based and wireless channels [6]. Furthermore, the presentation will be restricted to real valued schemes and channels (an extension to complex valued schemes is straightforward).

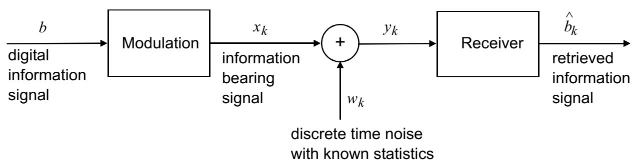

Figure 1: Coded modulation block scheme.

We consider a setup as in Fig. 1, where the transmitter realizes a coded modulation (modulation with memory). We postpone the discussion of the receiver to Sec. 3 and concentrate on the coded modulation realized by controlling a suitable chaotic system by small perturbations. There is some evidence that such control is an efficient means of controlling the symbolic sequence of chaotic systems [3] such as the Lorentz system (a continuous-time chaotic system).

We first develop the coded modulation from a point of view of chaotic systems involving iterations of maps[1]. We then show that the setup also permits the point of view of a trellis coded modulation (based on a convolutional coder and a suitable signal constellation [6]).

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[1] There is no spreading - the speed of b_k and of x_k is the same.

2.1. Map view

Conceptually, we will exploit the fact that the symbolic sequence of chaotic systems can be controlled by a suitably designed control procedure which only injects small perturbation into the feedback of the chaotic system. It has been shown recently [5] that chaos based communications which exploit the symbolic sequence emitted by a chaotic system for transmitting their payload information tend to exploit the capacity of a given communication channel more efficiently. Therefore, the modulations constructed in this paper are such that the information sequence $\boldsymbol{b} = [b_1, \ldots, b_k, \ldots]$ is transformed into a sequence of real values $\boldsymbol{x} = [x_1, \ldots, x_k, \ldots]$ such that $\boldsymbol{b}$ coincides with the symbolic sequence of $\boldsymbol{x}$ up to a shift (coding delay).

To be precise, we define the symbolic sequence $\boldsymbol{s} = [s_1, \ldots, s_k, \ldots]$ associated with $\boldsymbol{x}$ by means of a suitable quantization $\mathbb{R} \to \{0, 1, \ldots, M-1\}$, for the binary case $(M = 2)$ being a simple threshold. Without loss of generality we assume $x_k \in \mathbb{I} = [0, 1]$.

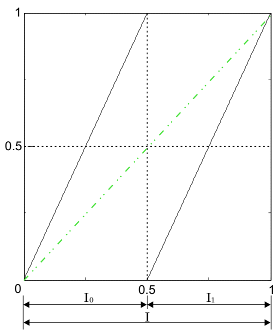

Figure 2: The bernoulli shift map.

Let us start by investigating the coded modulation obtained by controlling the Bernoulli shift map (Fig. 2)

$$x_{k+1} = f(x_k) \equiv 2x_k \ mod\ 1. \qquad (1)$$

We will later extend the class of maps by including maps that are topologically conjugate to the Bernoulli shift.

First note that the Bernoulli shift, as all piecewise linear Markov maps [7], admits a *natural* symbolic sequence associated with the index of the intervals $\mathbb{I}_{0,1}$ on which the map is linear[2].

For the Bernoulli shift, the state x_k can be represented as a binary expansion:

$$x_k = 0.s_k s_{k+1} s_{k+2} \ldots \equiv \sum_{i=0}^{\infty} 2^{-(i+1)} s_{k+i}. \qquad (2)$$

[2]It is worth pointing out that such *natural* symbolic sequence models the *information production* of the associated map completely (the Kolmogorov-Sinai [6] entropy of the map is equal to the Shannon entropy of the symbolic sequence considered as a random sequence).

where each of the bits s_i is either 0 or 1. We obtain the next state x_{k+1} by shifting the position of the bits of x_k to the left (multiplication by 2) and setting to zero the integer digit (operation of modulo 1). Observe that in this representation, the symbol in the symbolic sequence which is associated to x_k is given directly by s_k. Therefore, we can control the symbolic sequence by small perturbations (perturbing x_k at the Q-th digit will manifest itself in the symbolic sequence with a coding delay of Q). For simplicity, x_k will simply be quantized to $Q+1$ digits since any additional digits would be removed by the control anyway. We can therefore rewrite the signal x_k in terms of the bits $\boldsymbol{b}$ to be transmitted, made causal by a timeshift of Q, assuming a small perturbation control of amplitude $\pm q/2$ with $q = 2^{-Q}$:

$$x_k = \sum_{i=0}^{Q} 2^{-(i+1)} b_{k+i-Q} + q/2, \qquad (3)$$

where the additive term $q/2$ ensures the symmetry of the perturbation. This modulation rule can now be represented as a small perturbation control by rewriting (1) as in terms of (3)

$$x_{k+1} = f(x_k) + q \underbrace{(b_{k+1} - 1/2)}_{\pm 1/2} \qquad (4)$$

as illustrated in Fig. 3. Thus, in terms of a controlled chaotic system, the modulation is best described by (4). It is easy to prove that (3) leaves a discrete set of points in $[0, 1]$ invariant. However, by focusing on (1) and (3) it turns out that another description is possible, one that is more familiar to communication engineers, that of a trellis coded modulation involving a convolutional coder.

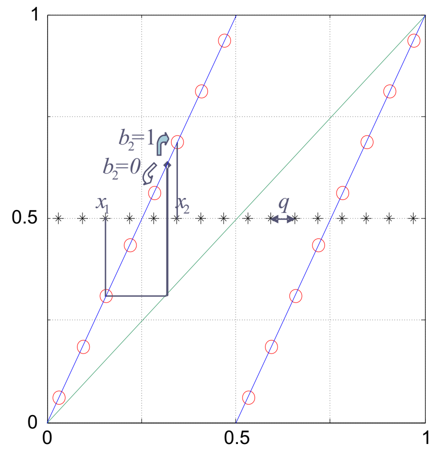

Figure 3: Chaotic system view.

2.2. Convolutional coder view

The above scheme described by (4) can be represented by means of a convolutional coder of rate $1/Q$ (Fig. 4, left part of the dotted line) and a signal constellation realized by a weighted sum (coefficients $c_i = 2^{-(i+1)}$). We emphasize that the signal x_k takes one of 2^Q discrete values uniformly spaced in the interval $[0, 1]$.

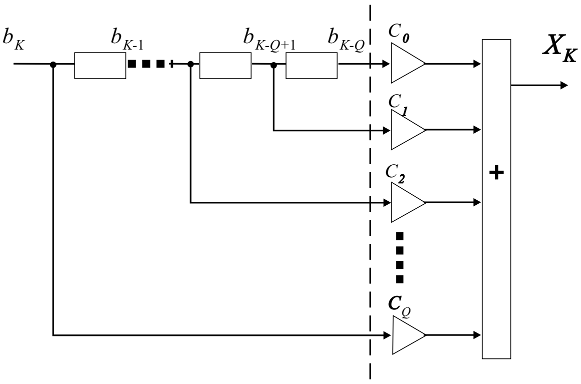

Figure 4: Convolutional coder view.

Using (4), the representation of the transmitter as given in Fig. 4 is clearly equivalent to the *map view* given before. In the representation of Fig. 4, it becomes obvious that the modulation scheme has indeed the form of a trellis coded modulation, consisting of a (convolutional) coder and a signal constellation[3].

2.3. Generalization: linear coder and corresponding Maps

It should be emphasized that the convolutional code used in Fig. 4 is a trivial one which does not yield optimal performance. To improve the performance, we extend the system to include a non-trivial linear convolutional code described by a binary matrix T. The scheme extended in such a way is shown in Fig. 5 where $\oplus$ is the $GF(2)$ sum (XOR operation).

The approach of introducing the code described by T is somewhat similar to the approach in [2] where instead the signal constellation was permuted in order to improve the performance, giving up the simplicity of the weighted sum generating the signal constellation. The authors prefer the present approach since it effectively avoids a problem of [2] which is the equivalence in terms of performance of a large number of permutations of the signal constellation. Note that it is reasonable to require T to be of full rank over $GF(2)$, in order to assure that it corresponds to an equivalent permutation. Furthermore,

[3] For our example, the signal constellation is a PAM one, that is, on the real line. However, other constellations, such as a PSK where all signal points are located on the unit circle can be introduced in a straightforward manner.

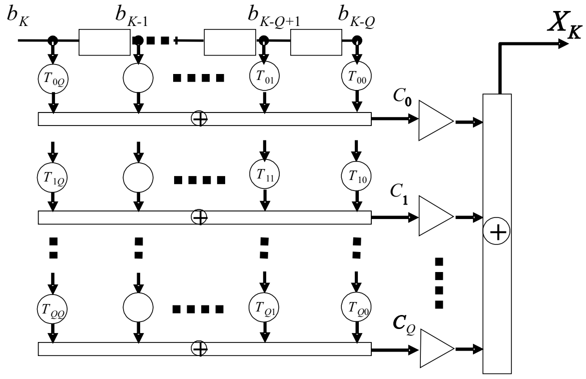

Figure 5: Convolutional coder view with matrix T.

due to the linearity of the code described by T, observe that in principle not all permutations are possible.

2.3.1. Some example of maps

The linear convolutional code introduced above would be of no value if it could not be included in the equivalence relation between the *map view* and the *convolutional coder view.* Therefore we are in fact looking for $Q \times Q$ matrices T that can be extended in a systematic way as $Q \to \infty$ such that the the equivalent map converges to an interesting function. For the purpose of this paper we give two examples:

$$T_{\text{shift}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ & & \\ 0 & 0 & \ddots \end{bmatrix} \qquad T_{\text{tent}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ & & \\ 1 & 1 & \ddots \end{bmatrix} \tag{5}$$

which can be readily extended to any matrix dimension. It is not difficult to see that, as $Q \to \infty$, the scheme using T_{shift} will converge to the Bernoulli shift map, while using T_{tent} will yield the tent map.

3. Decoding Coded Modulations

3.1. Viterbi decoder algorithm

For the decoding (receiver) side, we can leverage the existing knowledge about trellis coded modulations.

The method of choice for decoding trellis coded modulations is the Viterbi algorithm [1], which made its first appearance in the coding literature in a paper written by A. J. Viterbi in 1967. The Viterbi algorithm is a maximum-likelihood decoding procedure for convolutional codes. For the sake of conciseness, we present only the most important concepts of the Viterbi algorithm, the trellis diagram (shown in Fig. 6). The Viterbi algorithm performs *maximum a posteriori* decoding with linear complexity in the number of transmitted bits. It

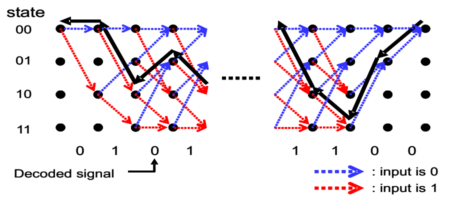

Figure 6: Trellis for Q=3. Each node corresponds to a distinct state at given time, and each arrow represents a transition to some new state at the next instant of time.

can be adapted to different channel noise statistics by modifying the branch metrics computed at each step. For more details see [1, 6].

3.2. Sliding window decoder algorithm

Another approach to decoding which minimizes the complexity of the decoder is based on the exponentially decreasing contributions which each of matrix T's lines gives towards the modulated signal. If, additionally, T has a particular structure, that is, lower triangular, it is easy to see that for decoding b_{k-Q}, only $R < Q+1$ signal samples need to be taken into account when $c_i, i \geq R$ are assumed to be small (negligible) with respect to the noise in the channel. One can formalize this concept by computing a decision variable based on the appropriate likelihood ratio.

$$D_k = \frac{p(b_k = 1|y_k, \ldots, y_{k+R-1})}{p(b_k = 0|y_k, \ldots, y_{k+R-1})} \lessgtr 1 \qquad (6)$$

which can be computed by marginalizing with respect to the noise and all other involved bits (independent!).

4. Performance

In this section we compare the performance of our modulation scheme for different matrices T and for different decoding methods (Viterbi and sliding window). Fig. 7 shows the BER performance for the considered variants with $Q = 5$, for the previously defined matrices T_{shift}, T_{tent} and an optimized matrix T_{opt} under Viterbi decoding, as well as T_{shift} under sliding window decoding with window length $R = 1$ and $R = 2$. The BPSK (binary phase shift keying) curve is also plotted for reference purposes. Observe the good performance of T_{shift} and T_{opt} under Viterbi decoding, but also under sliding window decoding for $R = 2$ at a much lower complexity.

5. Conclusions

The paper has demonstrated a coded modulation method based on a chaotic system controlled by small perturbation. For the chosen Bernoulli shift map, it has been shown that such a system is equivalent to a trellis coded modulation method which is well known in the communications literature. This equivalence allows the use of all the known tools of communications theory in this context, and, at the same time, also builds a bridge to modulation schemes based on chaotic systems. It has been shown how under certain conditions, the convolutional codes appearing in the equivalent trellis coded modulation can be related to simple non-linear, chaotic maps. The authors believe that an extension of this technique to better (convolutional etc.) codes, and the extension of the map to multiple dimensions will lead to an interesting view on coded modulations and potentially to an alternative implementation methodology, based on small perturbation control of suitable chaotic systems.

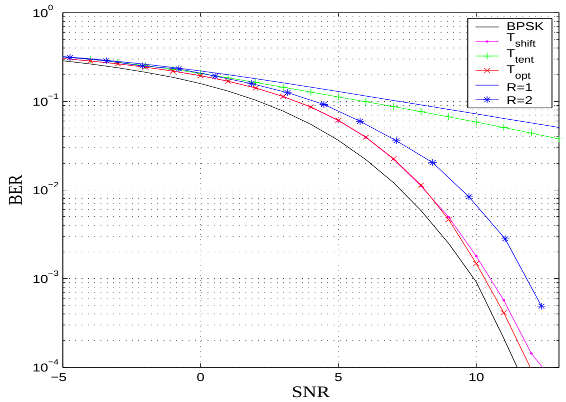

Figure 7: BER Performance for Q=5.

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