# On Basin Boundary of the Extended BVP Oscillator

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**Abstract**

We study a conspicuous behavior called jack-in-the-box in the extended BVP oscillator. In the three-dimensional system, the orbit started from an arbitrary initial point makes turns around the origin and gradually approach it, however, it is finally captured by one of stable equilibria located far away from the origin. Furthermore, we find out that the basin of attractions for these two sinks forms blurred boundary. In other words, even the system is bounded and there only exist two sinks as the final state, however, we cannot locate the initial point flowing into the specified sink. We demonstrate this phenomenon by numerical simulations.

## 1. Introduction

Bonhöffer-van der Pol (BVP) oscillator has been introduced from excitatory activity of the dynamical system described by the Hodgkin-Huxley (HH) equation. Many coupled BVP systems exhibit complex behavior[1, 2, 3]. We previously reported the dynamical behavior of an extended version of BVP oscillator[4] which is only attached an capacitor to the original BVP oscillator. Figure 1 shows the circuit configuration. This circuits is three-dimensional autonomous system and it has period-doubling cascades to chaos. A rich variety of chaotic attractors like Chua's circuit are found.

In this paper, we focus our eyes on the jack-in-the-box (abbr. JITB) phenomenon, which also has been briefly introduced in Ref. [4]. Firstly we show some examples of JITB phenomenon and give a heuristic explanation about generation of this phenomenon. Secondly we study basin boundaries of two sinks. It is marvelous that a basin of attractions of them are lost their deterministic structure. In other words, we conclude that we cannot predict which sink is chosen as a final position. Some numerical simulations are shown as an evidence.

## 2. The Extended BVP Oscillator

The normalized equation of the extended BVP oscillator is written as follows:

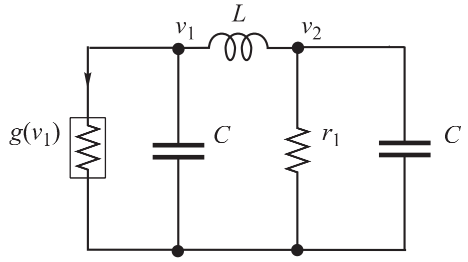



Figure 1: The extended BVP oscillator.

$$\frac{dx}{dt} = -z + \tanh \gamma x, \quad \frac{dy}{dt} = z - \delta y, \quad \frac{dz}{dt} = x - y \quad (1)$$

where, $\gamma$ and $\delta$ are parameters.

Figure 2 shows a bifurcation diagram of equilibria and periodic solutions in $\gamma$-$\delta$ plane. $d$, $h$ are pitchfork and Hopf bifurcation of the equilibrium, $G$, $I$, $Pf$ show tangent, period-doubling and pitchfork bifurcation for a limit cycle, respectively. Subscripts of these symbols are for enumeration. This figure is obtained by calculating bifurcation conditions with the suitable Poincaré mapping. There are many codimension-$n$ bifurcation sets in this figure, especially, $(\gamma, \delta) = (0, 0)$ becomes a very special degenerate parameter value.

In area (A) to (D) in Fig. 2, there exist at least one limit cycle in large. For example, in area (B), there exist a stable, two unstable limit cycles, two sinks, and saddle origin. In area (E), all limit cycles are disappeared by touching tangent bifurcation set $G_2$, and there are two symmetrical sinks $C^{\pm}$.

## 3. Jack-In-The-Box Phenomenon

Here we briefly summarize the JITB phenomenon[4]. Let us assume the state space of the extended BVP oscillator and fix the parameters in area (E) in Fig. 2. If you choose an arbitrary big initial point $(x_0, y_0, z_0) \in \boldsymbol{R}^3$ except for the domain stated later, firstly the orbit starting this point approaches the origin $O$. In most cases,

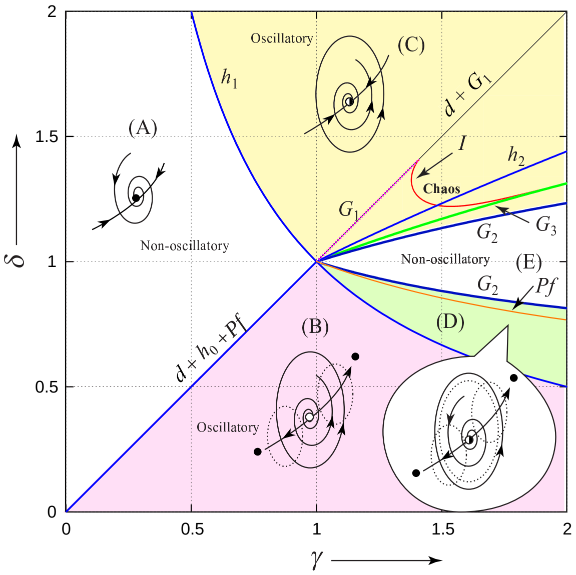



Figure 2: Bifurcation diagram of Eq. (1)

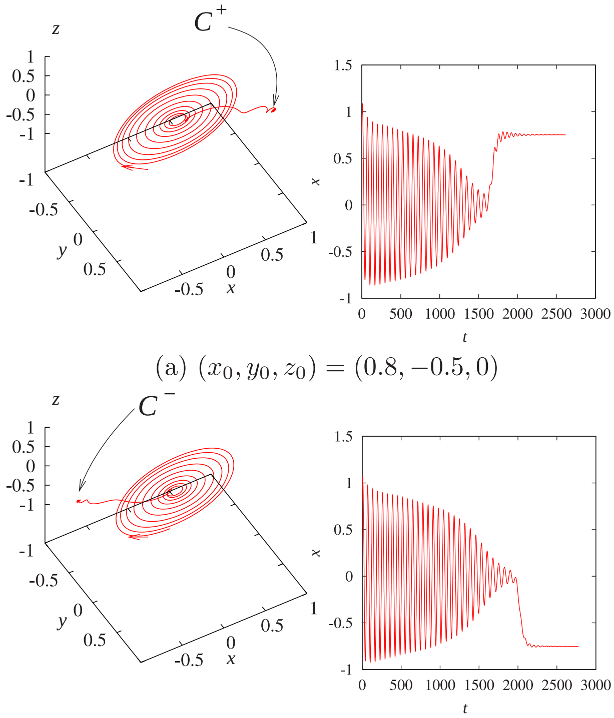
the orbit makes turns around $O$ like a damped harmonic oscillation, i.e., the orbit seems to be shrink to $O$. Remind that the limit cycle is disappeared by the tangent bifurcation in this parameter region, but the wreckage of it exists. We call it a pseudo limit cycle. At this situation, it can be regarded that the orbit is captured by the pseudo limit cycle having a wide range of basin of attraction temporary and then the orbit slides on the stable manifold $W^s$ of the $O$. However, the origin is a saddle, thus if the orbit reaches the linearized region of the origin, the orbit suddenly takes turn and escape to a sink along an unstable manifold of $O$.

Figure 3 illustrates numerical simulation of the JITB phenomenon, and Fig. 4 shows a schematic diagram. This phenomenon is also confirmed in a physical electric implementation[4].

There also exist a domain in which the orbit does not behave the JITB. In this domain, the orbit directly approaches $O$. It is mentioned in the next section.

## 4. Domain of Jack-In-The-Box

We discuss here the basin of attractions for $C^{\pm}$. The basin boundary of two sinks in this system are distinguished by the stable manifold $W^s$ of the origin. Let us remember the case of Lorenz equation. As is well known, the stable manifold forms a surface in $\boldsymbol{R}^3$ and its forms like a croissante, therefore, the surface is continuous and does not intersect itself. A basin of attraction for a sink is continuous, in other words, there must exist a line in $\boldsymbol{R}^3$ which can directly connect arbitrary chosen two points in a basin.




(a) $(x_0, y_0, z_0) = (0.8, -0.5, 0)$

(d) $(x_0, y_0, z_0) = (0.8, -0.6, 0)$

Figure 3: Phase portrait. $\gamma = 1.3$, $\delta = 1$.

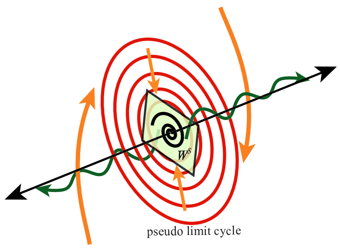



Figure 4: The model of JITB phenomenon.

For Eq.(1), since the origin has stable complex conjugate eigenvalues, it is difficult to obtain the stable manifold as a surface model. (It is not easy to have 3D structure of the manifold even if the origin has real eigenvalues, see Ref. [5]. Now we grasp the whole structure of the basins, we fix $z_0 = 0$ (see Fig. 5), and distinguish which the orbit starting $(x_0, y_0, 0)$ finally takes $C^+$ or $C^-$. We color them in the whole $x$-$y$ plane.

Firstly we choose the parameters as $\gamma = 1.3$, $\delta = 1$ in area (E) in Fig. 2. The black and white regions show sets of initial points converging to $C^-$ and $C^+$, respectively. The edge of both region indicates a basin boundary and it smoothly split two regions. It is a reasonable result which is also prognosticated from the result of the original (second-dimensional) BVP equation. While, these figures loses information of the time. It is experimentally

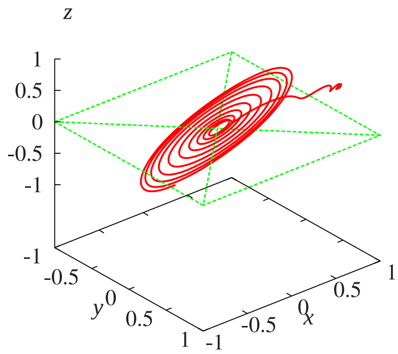



Figure 5: $z = 0$ plane and the JITB orbit.

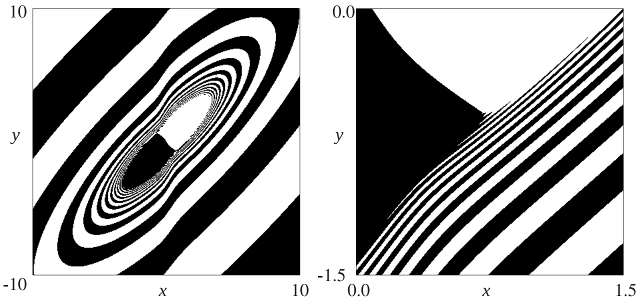
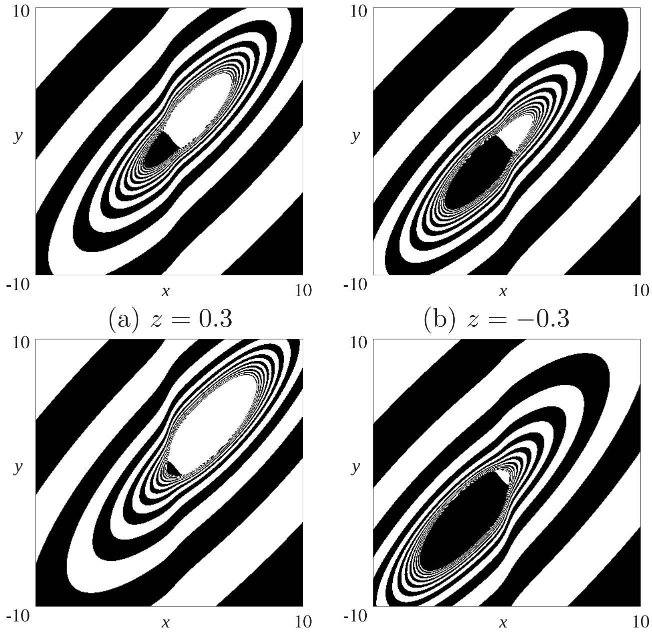



Figure 6: Basin boundary of extended BVP oscillator. Left: global range, Right: magnification.




(a) $z = 0.3$

(b) $z = -0.3$

(c) $z = 0.8$

(d) $z = -0.8$

Figure 7: Basin boundaries with several values of $z_0$.

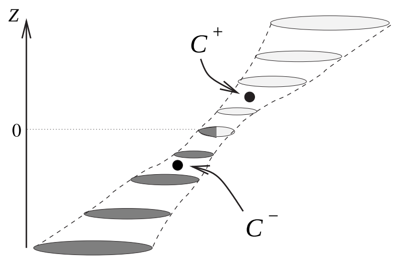



Figure 8: The domain of the non-JITB.

clarified that an oval region in the center part of this figure visualize a couple of the initial point sets where the JITB phenomenon is not occurred.

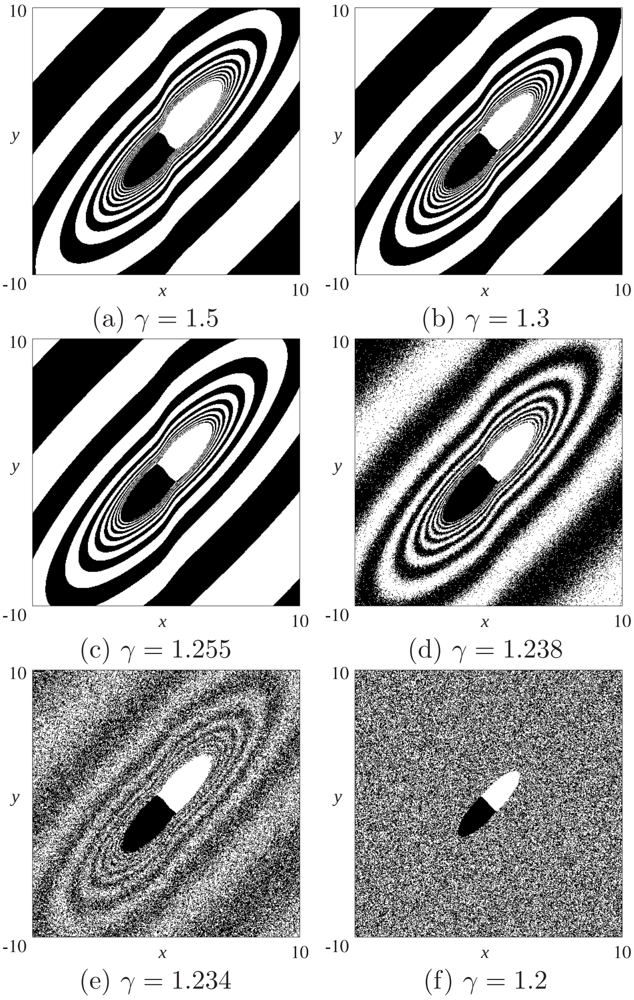
Probably the stable manifold forms a surface, thus to imagine its structure, we have to collect basin boundaries with different values of $z_0$ and bind them. Figure 7 show changing basin boundary with some values of $z_0$. From these pictures, the stable manifold also forms lile a croissante, and it is topologically equivalent to the Lorenz system.

Figure. 8 shows a domain of non-JITB constructed by ovals in Figs. 7 and the domain forms a hyperboloid of one sheet in $\boldsymbol{R}^3$. If we place an initial point inside of the hyperboloid, the trajectory approaches the sink directly (without approaching the origin). Conversely, if we place an initial point outside of this hyperboloid (a complement of the domain of non-JITB), the trajectory is captured by a pseudo limit cycle and then it must indicate the JITB phenomenon.

## 5. Blurred Basin Boundary

Fig. 9 illustrates basin boundaries as $\gamma$ changes. In the range $\gamma > 1.255$, there is no special change for the boundary. However, at $\gamma = 1.234$, the boundary is gradually blurred as $\gamma$ changes. When we choose $\gamma = 1.2$, the boundary is completely lost so that we cannot predict the final state regardless there exist only two states. We also cannot locate the initial point flowing into the specified sink surely.

It is generally received that the complex basin boundary is brought by fractal basin boundaries for invertible systems including the systems described by ODEs. Thus even if the system has a fractal basin boundary, the basins are continuously connected in the state space, i.e., there must exist a line between two points in the same basin. However, we cannot find any fractal structure for the extended BVP oscillator. In this system, basin boundary is blurred suddenly without forming fractal basin. Although we enlarge the blurred boundary over and over, we cannot find any structure. Figure 10 shows enlargement near the interface part of the oval. Clearly we can see that the shape of the boundary is almost

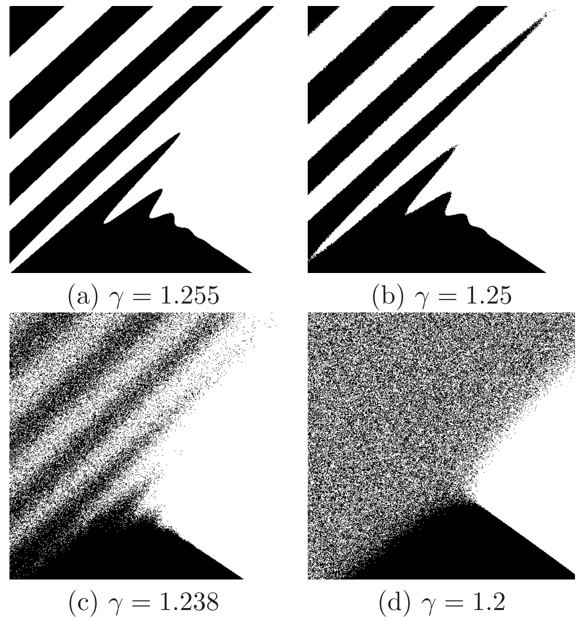



(a) $\gamma = 1.5$ (b) $\gamma = 1.3$

(c) $\gamma = 1.255$ (d) $\gamma = 1.238$

(e) $\gamma = 1.234$ (f) $\gamma = 1.2$

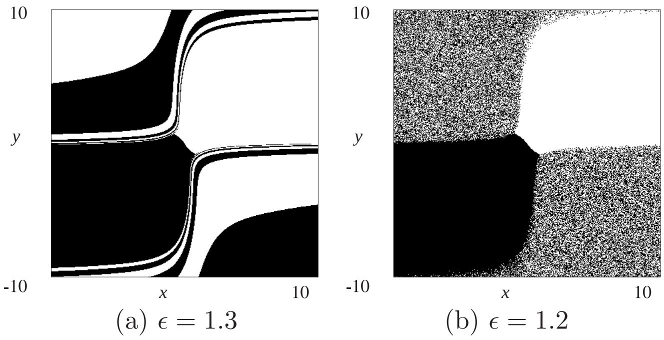
Figure 9: The blurred basin boundaries.



(a) $\gamma = 1.255$ (b) $\gamma = 1.25$

(c) $\gamma = 1.238$ (d) $\gamma = 1.2$

Figure 10: The enlarged figures of Fig. 9.




(a) $\epsilon = 1.3$ (b) $\epsilon = 1.2$

Figure 11: Basin boundary of $\epsilon(1 - x^2/3)x$.

fixed and accumulation of the boundary is not appeared. Therefore, the situation depicted in Fig.10(d) is not caused from fractal basin boundary. We can say that basin boundary loses its deterministic structure.

Note that these phenomena are not special for the characteristics of $g(v)$. In fact if we choose $\epsilon(1 - x^2/3)x$ instead of $\tanh \gamma x$, topologically similar structure is observed, see Fig. 11. Thus we can conclude experimentally that the blurred basin boundary is caused by the third-order nonlinearity.

## 6. Conclusions

In this paper we investigated the jack-in-the-box phenomenon and its complex basin boundary. As far as the authors know, these physical properties have not reported yet and also we have not found similar phenomenon in other three-dimensional systems, such as Chua's circuit. In future, we would like to clarify what can destroy the basin boundary.

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