Global Representation of Stationary Solutions for Impact Oscillator

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Summary

1. System Discription in Pseudo-Feedback Form

In this paper, we consider global functional structure of the stationary solution for forced impact oscillator. The nondimensional equations of motions of this system are given as follows.

$$\ddot{x} + 2\zeta\dot{x} + \beta x = \eta\cos(\omega t); \quad x < \gamma \tag{1a}$$

$$\dot{x}(t^+) = -r\dot{x}(t^-); \quad x = \gamma \quad (0 \le r \le 1) \tag{1b}$$

where ζ is a damping factor, β is a stiffness, and $\eta\cos\omega t$ is external force respectively. Mass motion are constrained by a rigid wall, and mass bounce off this wall at the impact velocity multiplied by the coefficient of restitution $(0 \le r \le 1)$ at impact instance. This system is the most fundamental piecewise linear system, and a great deal of effort has been performed to investigate its chaotic behaviour and bifurcation analysis by many reserachers. The general solution of this system could be represented globaly in self reference point of view has been shown in our previous work[1]. Moreover, we proposed the exact derivation mothod of stationary solutions from this global general solutions by parametrizing only the inplicit impact times[2, 3]. In this paper, we propose new exact derivation method of stationary solutions for impact oscillator (1), and compare the result derived by this method and that of former proposed one. If an impulse force generated by a strong impact nonlinearity is regarded as an apparent external force on the system (1) and transposed to the right hand side of the equation of motion, the following equivalent equation can be obtained.

$$\ddot{x} + 2\zeta\dot{x} + \beta x = -\sum_{t_i \in Q}(1+r)\dot{x}(t_i^-)\delta(t-t_i) + \eta\cos\omega t \qquad ; x \le \gamma \tag{2}$$

Here Q is the set of impact times $t_i (i \in Z)$ for which the solution becomes $x(t_i) = \gamma$. Let us call this strictly linearized formulation "pseudo-feedback form"[1]. Since the impulse force on the right hand side contains the impact velocity which is equal to the derivative of the solution to be obtained, this is a kind of implicit equation where the solution to be obtained returns itself. The state transition matrix $\boldsymbol{\mathscr{T}}(t)$ and the input response function vector $\boldsymbol{\mathscr{S}}(t)$ for linear system are defined as follows.

$$\boldsymbol{\mathscr{T}}(t) \equiv \begin{bmatrix} \mathscr{T}(t) & \mathscr{F}(t) \\ \dot{\mathscr{T}}(t) & \dot{\mathscr{F}}(t) \end{bmatrix}, \quad \boldsymbol{\mathscr{S}}(t) \equiv \begin{bmatrix} \mathscr{S}(t) \\ \dot{\mathscr{S}}(t) \end{bmatrix} \tag{3}$$

The elements of the matrix $\boldsymbol{\mathscr{T}}(t)$ is classified in β-ζ parameter space as shown in table 1. The input response function $\mathscr{S}(t)$ can be represented as following.

$$\mathscr{S}(t) \equiv \frac{\eta}{\sqrt{(\beta-\omega^2)^2 + (2\zeta\omega)^2}} \cos\left(\omega t + \tan^{-1}\frac{-2\zeta\omega}{\beta-\omega^2}\right) \tag{4}$$

The state vector is denoted by $\boldsymbol{\mathscr{X}}(t) \equiv [\mathscr{X}(t), \dot{\mathscr{X}}(t)]^t$ where the elements are displacement and velocity of stationary solution, and $[\ \ ]^t$ is a notation of matrix transpose.

Table 1 State transition functions $\mathscr{T}(t)$ and $\mathscr{F}(t)$ for $\beta - \zeta$ parameter space

parameters	$\mathscr{T}(t)$	$\mathscr{F}(t)$
$\beta > \zeta^2, \beta \neq 0$	$e^{-\zeta t}\big(\cos\Omega^+ t + \frac{\zeta}{\Omega^+}\sin\Omega^+ t\big)$	$e^{-\zeta t}\frac{1}{\Omega^+}\sin\Omega^+ t$
$\beta < \zeta^2, \beta \neq 0$	$e^{-\zeta t}\big(\cosh\Omega^- t + \frac{\zeta}{\Omega^-}\sinh\Omega^- t\big)$	$e^{-\zeta t}\frac{1}{\Omega^-}\sinh\Omega^- t$
$\beta = \zeta^2, \beta \neq 0$	$e^{-\zeta t}(1+\zeta t)$	$e^{-\zeta t} t$
$\beta = 0, \zeta \neq 0$	1	$\frac{1-e^{-2\zeta t}}{2\zeta}$
$\beta = \zeta = 0$	1	t

2. Stationary Solution for Impact Oscillator

The general solution of impact oscillator (1), not accompanied by sticking, emanated from the initial time t_j^- to the infinite past and infinite future can be represented as the following equations respectively.

$$
\begin{aligned}
&u(t-t_j)\boldsymbol{\mathscr{X}}_\infty(t;\boldsymbol{\mathscr{X}}_\infty(t_j^-)) \\
&\quad = u(t-t_j)[\boldsymbol{\mathscr{T}}(t-t_j)\{\boldsymbol{\mathscr{X}}_\infty(t_j^-;\boldsymbol{\mathscr{X}}_\infty(t_j^-)) - \boldsymbol{\mathscr{S}}(t_j)\} + \boldsymbol{\mathscr{S}}(t)] - (1+r)\sum_{i=j}^{+\infty} u(t-t_i)\boldsymbol{\mathscr{F}}(t-t_i)\dot{\mathscr{X}}_\infty(t_i^-)
\end{aligned}
\tag{5a}
$$

$$
\begin{aligned}
&u(-(t-t_j))\boldsymbol{\mathscr{X}}_\infty(t;\boldsymbol{\mathscr{X}}_\infty(t_j^-)) = u(-(t-t_j))[\boldsymbol{\mathscr{T}}(t-t_j)\{\boldsymbol{\mathscr{X}}_\infty(t_j^-;\boldsymbol{\mathscr{X}}_\infty(t_j^-)) - \boldsymbol{\mathscr{S}}(t_j)\} + \boldsymbol{\mathscr{S}}(t)] \\
&\quad + (1+r)\sum_{i=-\infty}^{j-1} u(-(t-t_i))\boldsymbol{\mathscr{F}}(t-t_i)\dot{\mathscr{X}}_\infty(t_i^-)
\end{aligned}
\tag{5b}
$$

These equations can be combined and rewritten into backward and forward transition form in entire time interval as follows.

$$
\begin{aligned}
&\boldsymbol{\mathscr{X}}_\infty(t;\boldsymbol{\mathscr{X}}_\infty(t_j^-)) = \boldsymbol{\mathscr{T}}(t-t_j)[\boldsymbol{\mathscr{X}}_\infty(t_j^-;\boldsymbol{\mathscr{X}}_\infty(t_j^-)) - \boldsymbol{\mathscr{S}}(t_j)] + \boldsymbol{\mathscr{S}}(t) \\
&\begin{cases}
+(1+r)\displaystyle\sum_{i=-\infty}^{+\infty} u(-(t-t_i))\boldsymbol{\mathscr{F}}(t-t_i)\dot{\mathscr{X}}_\infty(t_i^-) - (1+r)\sum_{i=j}^{+\infty}\boldsymbol{\mathscr{F}}(t-t_i)\dot{\mathscr{X}}_\infty(t_i^-) \\
-(1+r)\displaystyle\sum_{i=-\infty}^{+\infty} u(t-t_i)\boldsymbol{\mathscr{F}}(t-t_i)\dot{\mathscr{X}}_\infty(t_i^-) + (1+r)\sum_{i=-\infty}^{j-1}\boldsymbol{\mathscr{F}}(t-t_i)\dot{\mathscr{X}}_\infty(t_i^-)
\end{cases}
\end{aligned}
\tag{6}
$$

The value of the summation in equation (6) should be bounded for stationary solution. Namely, for parameter region $\beta \neq 0$, the first and fourth term in the right hand side in equation (6) will be cancel each other. On the other hand, for parameter region $\beta = 0$, the value of the summation of the first and fourth term in the right hand side in equation (6) is bounded constant value. Therefore the initial values of stationary solutions are saticefied with following relation.

$$
\boldsymbol{\mathscr{X}}_\infty(t_j^-;\boldsymbol{\mathscr{X}}_\infty(t_j^-)) = \boldsymbol{\mathscr{C}} + \boldsymbol{\mathscr{S}}(t_j) +
\begin{cases}
(1+r)\displaystyle\sum_{i=j}^{+\infty}\boldsymbol{\mathscr{F}}(t_j-t_i)\dot{\mathscr{X}}_\infty(t_i^-) \\
-(1+r)\displaystyle\sum_{i=-\infty}^{j-1}\boldsymbol{\mathscr{F}}(t_j-t_i)\dot{\mathscr{X}}_\infty(t_i^-)
\end{cases}
\tag{7}
$$

where $\boldsymbol{\mathscr{C}} \equiv [\mathscr{C}, 0]^t$ is a vector of an indefinite value related to the initial conditions for $\beta = 0$ parameter region . If these initial values of the stationary solution (7) are substituted for a general solution (6) that remains true from the infinite past to the infinite future, the following global expression of the stationary solution can be obtained.

$$\begin{aligned}\boldsymbol{\mathscr{X}}_{\infty}(t) &= \boldsymbol{\mathscr{C}} + \boldsymbol{\mathscr{S}}(t) \pm (1+r)\sum_{i=-\infty}^{+\infty} u(\mp(t-t_i))\boldsymbol{\mathscr{F}}(t-t_i)\dot{\mathscr{X}}_{\infty}(t_i^-) \\ &\equiv \boldsymbol{\mathscr{C}} + \boldsymbol{\mathscr{S}}(t) + \boldsymbol{\mathscr{H}}_{\infty}(t)\end{aligned} \tag{8}$$

Since the two expressions (forward and backward transition processes) of the stationary solution (8) are equivalent, then the total amount of influences of nonlinearity of the stationary sotlution becomes 0.

$$\sum_{i=-\infty}^{+\infty} \boldsymbol{\mathscr{F}}(t-t_i)\dot{\mathscr{X}}_{\infty}(t_i^-) = \mathbf{0} \tag{9}$$

We call this formula "zero summation property". This indicates that the stationary solution is a kind of fixed point at which the stationary solution is such that the influences of nonlinearity are invariant. Meanwhile, if the taking limit of $t \to \pm\infty$ of the initial value of stationary solution (7), the following consequence can be obtained with regard to the states of the stationary solution in the infinite past and infinite future.

$$\boldsymbol{\mathscr{X}}_{\infty}(t_{\pm\infty}) = \boldsymbol{\mathscr{C}} + \boldsymbol{\mathscr{S}}(t_{\pm\infty}) \tag{10}$$

However, it looks somehow strange because this formura seems that the stationary solutions are coinside with that of linear system in the eternal future and the eternal past.

The derived representations (8) are natural extension of stationary solution for linear system into the case of nonlinear system. Besides, this representation perfectly prescribes all functional forms of the stationary solution include chaotic one.

3. New Method of Deriving Stationary Solution for Impact Oscillator

As indicated by the above result, the infinite past or infinite future can be interpreted as an ultimately linear region that is strictly linearized by pseudo feedback an infinite number of times. Therefore, we regard the states of infinite past and infinite future of stationary solutions as that of linear one. If we take limit $t \to \mp\infty$ of equation (5), formal representation of infinite past and infinite future of the ultimate linear region can be represented as follows.

$$\boldsymbol{\mathscr{X}}_{\infty}(t_{\mp\infty};\boldsymbol{\mathscr{X}}_{\infty}(t_j^-)) - \boldsymbol{\mathscr{S}}(t_{\mp\infty}) = \boldsymbol{\mathscr{T}}(t_{\mp\infty}-t_j)[\boldsymbol{\mathscr{X}}_{\infty}(t_j^-;\boldsymbol{\mathscr{X}}_{\infty}(t_j^-)) - \boldsymbol{\mathscr{S}}(t_j)] + \begin{cases} +(1+r)\displaystyle\sum_{i=-\infty}^{j-1} \boldsymbol{\mathscr{F}}(t_{-\infty}-t_i)\dot{\mathscr{X}}_{\infty}(t_i^-) \\ -(1+r)\displaystyle\sum_{i=j}^{+\infty} \boldsymbol{\mathscr{F}}(t_{+\infty}-t_i)\dot{\mathscr{X}}_{\infty}(t_i^-) \end{cases} \tag{11}$$

The equation (11) can be solved for $\boldsymbol{\mathscr{X}}_{\infty}(t_j^-) - \boldsymbol{\mathscr{S}}(t_j^-)$

$$\boldsymbol{\mathscr{X}}_{\infty}(t_j^-) - \boldsymbol{\mathscr{S}}(t_j) = \boldsymbol{\mathscr{C}} + \begin{cases} -(1+r)\displaystyle\sum_{i=-\infty}^{j-1} \boldsymbol{\mathscr{F}}(t_j-t_i)\dot{\mathscr{X}}_{\infty}(t_i^-) \\ +(1+r)\displaystyle\sum_{i=j}^{+\infty} \boldsymbol{\mathscr{F}}(t_j-t_i)\dot{\mathscr{X}}_{\infty}(t_i^-) \end{cases} \tag{12}$$

where, we use following relation for $\beta = 0$ parameter region.

$$\boldsymbol{\mathscr{T}}(t)\boldsymbol{\mathscr{C}} = \boldsymbol{\mathscr{C}} \tag{13}$$

By substituting this initial value (12) into the initial value of general solution (5), we can derived following equations.

$$u(\mp(t-t_j))\boldsymbol{\mathscr{X}}_{\infty}(t;\boldsymbol{\mathscr{X}}_{\infty}(t_j^-)) = u(\mp(t-t_j))\left[\boldsymbol{\mathscr{C}} + \boldsymbol{\mathscr{S}}(t) \pm (1+r)\sum_{i=-\infty}^{+\infty} u(\mp(t-t_i))\boldsymbol{\mathscr{F}}(t-t_i)\dot{\mathscr{X}}_{\infty}(t_i^-)\right] \tag{14}$$

In this calculation process, we used zero summation property (9) and following relations.

$$\begin{aligned} u(-(t-t_j)) &= u(-(t-t_j))u(-(t-t_i)) \quad (t_j < t_i) \\ u(t-t_j) &= u(t-t_j)u(t-t_i) \quad (t_i < t_j) \end{aligned}$$

If taking limit $j \to \pm\infty$ of these half of forward and backward form of stationary solutions (14), stationary solutions in entire time interval (8) can be derived again.

Therefore, the stationary solutions are interpreted as solutions its period are infinity.

Stationary solutions are saticefied with nonhomogeneous linear equation which right hand side has self reference term.

$$[\boldsymbol{\mathscr{I}} - \mathscr{T}(t-t_j)][\boldsymbol{\mathscr{X}}_\infty(t_j^-;\boldsymbol{\mathscr{X}}_\infty(t_j^-)) - \boldsymbol{\mathscr{S}}(t_j)] = [\boldsymbol{\mathscr{I}} - \mathscr{T}(t-t_j)]\boldsymbol{\mathscr{H}}_\infty(t_j^-) \tag{15}$$

where $\boldsymbol{\mathscr{I}}$ is the identity matrix. The solutions of this equation is coinside with the initial values of the stationary solutions(12). If nonlinearity becomes zero, above nonhomogeneous linear equation deduce following homogenous linear equation because self reference term vanish.

$$[\boldsymbol{\mathscr{I}} - \mathscr{T}(t-t_j)][\boldsymbol{\mathscr{X}}_\infty(t_j^-;\boldsymbol{\mathscr{X}}_\infty(t_j^-)) - \boldsymbol{\mathscr{S}}(t_j)] = \mathbf{0} \tag{16}$$

If the above results are summarized, the following invariance becomes true for stationary solutions.

$$\boldsymbol{\mathscr{X}}_\infty(t) - \boldsymbol{\mathscr{S}}(t) - \boldsymbol{\mathscr{H}}_\infty(t) = \mathscr{T}(t-t_j)[\boldsymbol{\mathscr{X}}_\infty(t_j^-) - \boldsymbol{\mathscr{S}}(t_j^-) - \boldsymbol{\mathscr{H}}_\infty(t_j^-)] \equiv \boldsymbol{\mathscr{C}} \tag{17}$$

This invariance strongly suggests the stationary solution is a kind of fixed point that does not vary with the state transition matrix $\mathscr{T}(t)$.

4. Canonical Form of Stationary Solutions

Since equations (8) are recursive, with the impact velocity referring to itself at impact instance, so that these equations can be calculated by using substitution itself and expand. As a result of this procedures, we finaly derived the following expressions of the stationary solutions, which are not explicitly recursive.

$$\boldsymbol{\mathscr{X}}_\infty(t) = \boldsymbol{\mathscr{S}}(t) \pm (1+r)\sum_{k_1=-\infty}^{+\infty} u(\mp(t-t_{k_1}))\boldsymbol{\mathscr{F}}(t-t_{k_1})\Big[\sum_{i=1}^{+\infty}\Big\{\prod_{j=1}^{i-1}\sum_{\substack{k_{j+1}=-\infty \\ k_j > k_{j+1}}}^{k_j-1} \pm(1+r)\dot{\boldsymbol{\mathscr{F}}}(t_{k_j} - t_{k_{j+1}})\Big\}\dot{\mathscr{S}}(t_{k_i})\Big] \tag{18a}$$

$$= \boldsymbol{\mathscr{S}}(t) \pm (1+r)\sum_{k=-\infty}^{+\infty} u(\mp(t-t_k))\boldsymbol{\mathscr{F}}(t-t_k)\Big[\sum_{p=1}^{+\infty}\sum_{\substack{-\infty \le q_1 < \cdots < q_{p-1} \le k-1 \\ q_p = k}}\Big\{\prod_{j=1}^{p-1} \pm(1+r)\dot{\boldsymbol{\mathscr{F}}}(t_{q_{j+1}} - t_{q_j})\Big\}\dot{\mathscr{S}}(t_{q_1})\Big] \tag{18b}$$

References

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