An Improvement of Solution Path Tracing Algorithm Using Hyper-Ellipsoid —An Application to Computer Graphics—

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Abstract

Computer graphics (CG) gives influence on the various engineering fields, e.g., movie, virtual reality, web technology and medicine. To support their progress, it's important to develop rendering technique in CG. Among many rendering algorithms, the so-called ray tracing method is well-known, which can generate photo-realistic images. The images are generated by obtaining solutions of a set of nonlinear simultaneous equations, which is derived from each ray and all objects. In this paper, an improved solution path algorithm is proposed, which calculates the solution of the nonlinear equation for the ray tracing algorithm. By the proposed algorithm smaller number of the generated hyper-elipsoids achieves the shorter computation time than the previously proposed method.

1. Introduction

In the field of computer graphics (CG), it is well-known that the so-called ray tracing method[1] is one of the most effective algorithms which can visualize objects photo-realistically. In order to visualize a scene with some objects by the method, a solution of the nonlinear simultaneous equation is to be calculated, which is derived from a ray from the given viewpoint and the shape of the object. Therefore, it is seen that the ability of visualization in the method depends on solvability of the nonlinear equations by the applied numerical method. It is seen that the study of so-called homotopy method[2]-[5] has been made great strides, which globally calculates a solution of systems of nonlinear equations by tracing a solution path of the derived homotopy equation. The method has been applied to many important engineering problems. Related with accuracy, computational complexity and so on, its quality of computing has been discussed based on recent high performance computing technology[5].

The homotopy method is based on tracing path of $h(x,t) = 0$ which is connected with the solution of the objective nonlinear system of equation $f(x) = 0$. In the ray tracing, huge number of nonlinear equations must be solved, since the equations are formulated per pixel, ray and object. Therefore, more efficient and more reliable tracing algorithms have been studied. Among them an important algorithm is presented by Ushida[2], and Yamamura proposed a geometric method using hyper-sphere[3]. The method by Yamamura is practical so that and some important applications to circuit analysis for VLSI are reported[4]. Also they presented ability of the modified version of the method instead of hyper-sphere. Based on the concept, we proposed a method using hyper-ellipsoid instead of the sphere, which is set according to the solution path property obtained a priori[6][7].

In the previous work[6][7], a homotopy method by using hyper-ellipsoids instead of the hyper-sphere has been proposed, which calculates a solution of nonlinear systems of equation. By the method 2 types of the tracing algorithms have been proposed, which are the method of tracing using the affine transformation and the method of tracing without turning ellipsoids. Also we showed an application of the proposed algorithm to the ray tracing for a scene with free form surfaces. In this application, the proposed method achieves faster computarion than the method by using hyper-sphere. However, in the case for complicated objects, more efficiency is necessary.

In this paper, in order to reduce computational cost, we propose an improved tracing algorithm by hyper-ellipsoids, which do not intersect each other, but tangent. Then we apply our proposed method to visualization of soliton pulse by the ray tracing method. The proposed algorithm achieves the shorter computation than the previously proposed method.

2. Tracing Solution Path Algorithm Using Hyper-Ellipsoid

In this paper, we consider the following type of free form surface:

$$\begin{aligned} x &= f_1(u,v) \\ y &= f_2(u,v) \\ z &= f_3(u,v) \\ u_{\min} \leq\ & u \leq u_{\max} \\ v_{\min} \leq\ & v \leq v_{\max}. \end{aligned} \tag{1}$$

Here f_1, f_2 and f_3 are nonlinear and at least two times differentiable (abbreviated to C^2) with respect to u and v, respectively. We note here that this restriction is usually assumed due to smooth connection between the surfaces. Also a ray is defined by the given viewpoint (x_e, y_e, z_e) and direction (x_v, y_v, z_v) as:

$$\begin{aligned} x &= wx_v + x_e \\ y &= wy_v + y_e \\ z &= wz_v + z_e \\ w &> 0. \end{aligned} \tag{2}$$

Then, from Eqs.(1) and (2), a set of simultaneous nonlinear equation per pixel, ray and surface is derived as follows:

$$\begin{aligned} f_1(u,v) - wx_v - x_e &= 0 \\ f_2(u,v) - wy_v - y_e &= 0 \\ f_3(u,v) - wz_v - z_e &= 0. \end{aligned} \tag{3}$$

From Eq.(3) it is seen that the equation to be solved has 3 dimension. Of course, the dimension may be higher if the surface is defined as more complex form than Eq.(1). Therefore, in general, we consider the following n-dimensional nonlinear equation instead of Eq.(3):

$$f(x) = 0, \quad f : \overline{D} \subset \boldsymbol{R}^n \to \boldsymbol{R}^n, \tag{4}$$

where $\overline{D}$ is a closure of a nonempty open bounded set $D \subset \boldsymbol{R}^n$, $0 \in \boldsymbol{R}^n$ is a regular value of f, and f is C^2 with respect to $x \in \overline{D}$.

By choosing $x_0 \in \mathrm{Int}\overline{D}$ as a starting point, the following so-called homotopy equation is introduced:

$$h(x,t) = tf(x) + (1-t)g(x) = 0, \tag{5}$$

$$h : \overline{D} \times [0,1] \to \boldsymbol{R}^n.$$

Here Int denotes internal, $t \in [0,1]$. Also, $g(x) = 0, g : \overline{D} \subset \boldsymbol{R}^n$ is an auxiliary equation which is chosen as having an obvious solution $x_0 \in \overline{D}$.

We can assume that the solution set of Eq.(5) consists of continuously differentiable paths and/or loops. Since one can numerically trace such path from a known point on the path, a solution of Eq.(4) can be calculated if the path starting from the obvious solution $(x,t) = (x_0, 0)$ reaches $t = 1$ hyper-plane.

In order to trace the path from $(x_0, 0)$, we formulate an $(n+1)$ dimensional simultaneous equation between Eq.(5) and a hyper-ellipsoid, locating $t \geq 0$ and $(x_0, 0)$ is on its surface. If the hyper-ellipsoid is small enough, then it is seen that there is unique solution path within the hyper-ellipsoid, and that the equation has two solutions $(x_0, 0)$ and (x_1, t_1), which are obtained by a certain numerical method such as the Newton-Raphson method. It is obvious that on the solution path (x_1, t_1) is closer to the solution of Eq.(4) than $(x_0, 0)$. By repeating calculation of solution consisting of Eq.(5) and a hyper-ellipsoid with (x_i, t_i), $i = 0, 1, \cdots$, on its surface, a solution of Eq.(4) can be calculated if the path is connected with $t = 1$ hyper-plane. It is seen that efficiency of the proposed algorithm depends on size and shape of the hyper-ellipsoid.

In this paper, we define the following hyper-ellipsoid:

$$\begin{cases} \displaystyle\sum_{j=1}^{n} \frac{X_{j,k}^2}{la_{j,k}^2} + \frac{T_k^2}{lb_k^2} = 1 \\ \begin{pmatrix} X_{j,k} \\ T_k \end{pmatrix} = \begin{pmatrix} x_{j,k} - xc_{j,k-1} \\ t_k - tc_{k-1} \end{pmatrix} \end{cases} \tag{6}$$

Here $(xc_{j,k-1}, tc_{k-1})$ is the center of the given hyper-ellipsoid, and suppose that $(x^*_{j,k-1}, t^*_{k-1})$ satisfies Eq.(6). $(xc_{j,k-1}, tc_{k-1})$ is set on a line which passes along $(x^*_{j,k-1}, t^*_{k-1})$ and $(x^*_{j,k-2}, t^*_{k-2})$, which we define as follows:

$$\begin{pmatrix} xc_{j,k-1} \\ tc_{k-1} \end{pmatrix} = 2\begin{pmatrix} x^*_{j,k-1} \\ t^*_{k-1} \end{pmatrix} - \begin{pmatrix} x^*_{j,k-2} \\ t^*_{k-2} \end{pmatrix} \tag{7}$$

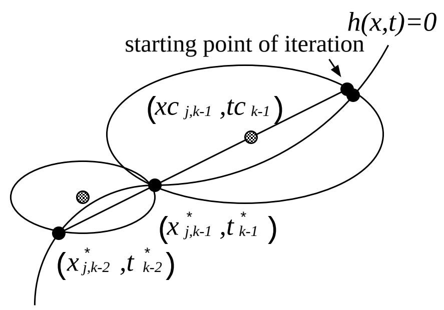

Figure 1: an outline of the proposed tracing algorithm

Also we define $la_{j,k}$ and lb_k as follows:

$$\left\{\begin{array}{rcl} la_{j,k} & = & \sqrt{n+1}\,\|xc_{j,k-1} - x^*_{j,k-1}\| \\ lb_k & = & \sqrt{n+1}\,\|tc_{k-1} - t^*_{k-1}\| \end{array}\right. \tag{8}$$

It is obviously seen that the derived nonlinear simultaneous equation of Eqs.(5) and (7) has at least two solutions, and one is $(x^*_{j,k-1}, t^*_{k-1})$. Consequently, the another solution must be obtained to reach the solution of Eq.(4). For the calculation, the Newton-Raphson method is introduced with a starting point $(3x^*_{j,k-1} - 2x^*_{j,k-2}, 3t^*_{k-1} - 2t^*_{k-2})$, where the point is located on the opposite position of $(x^*_{j,k-1}, t^*_{k-1})$ with respect to $(xc_{j,k-1}, t_{k-1})$. If the hyper-ellipsoid is small enough, then a solution of the simultaneous equation can be calculated. Repeating this procedure from $t = 0$ to $t = 1$ along the solution path, a solution of Eq.(5) is obtained (see Figure 1).

Here we compare the proposed method with the previously proposed method in Refs.[6],[7]. Eq.(9) and Figure 2 show a hyper-ellipsoid and outline of that method, respectively.

$$\left\{\begin{array}{l} \displaystyle\sum_{j=1}^{n} \frac{X^2_{j,k}}{la^2_{j,k}} + \frac{T^2_k}{lb^2_k} = 1 \\ \begin{pmatrix} X_{j,k} \\ T_k \end{pmatrix} = \begin{pmatrix} x_{j,k-1} - x^*_{j,k-2} \\ t_{k-1} - t^*_{k-2} \end{pmatrix} \\ la_{j,k} \;=\; \sqrt{n+1}\,\|x^*_{j,k-1} - x^*_{j,k-2}\| \\ lb_k \;=\; \sqrt{n+1}\,\|t^*_{k-1} - t^*_{k-2}\| \end{array}\right. \tag{9}$$

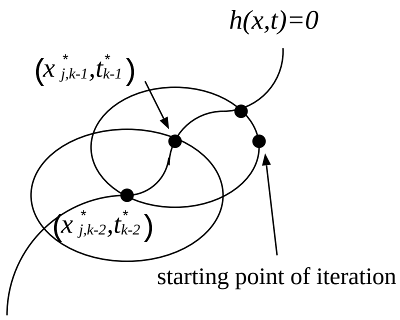

Figure 2: an outline of the tracing algorithm in [8],[9]

In the conventional method, a center of each ellipsoid is located at the previously obtained solution. As a consequence, the ellipsoids are overlapped each other. It causes larger number of the ellipsoids than the revised method is this paper from $t = 0$ to $t = 1$ along the solution path. Therefore it turns out that the proposed method in this paper is expected to get the advantage in computational cost.

3. An Application to Visualization of 2 Soliton Pulse

For the instance, we apply the proposed method to a visualization of a solution of the Korteweg-de Vrices (KdV) equation such that:

$$\begin{array}{l} \dfrac{\partial z}{\partial y} + 6z \cdot \dfrac{\partial z}{\partial x} + \dfrac{\partial^3 z}{\partial x^3} = 0, \\ z = 2 \cdot \frac{\partial^2}{\partial x^2}(\log f) \end{array} \tag{10}$$

We note that f is allowed in various forms. Among them,

$$f = 1 + e^{\kappa_1 x - \kappa_1^3 y} + e^{\kappa_2 x - \kappa_2^3 y} + \tfrac{(\kappa_1 - \kappa_2)^2}{(\kappa_1 + \kappa_2)^2} e^{(\kappa_1 + \kappa_2)x - (\kappa_1^3 + \kappa_2^3)y} \tag{11}$$

is known. In Eqs. (10) and (11), z is called 2-soliton solution.

When we substitute Eq.(2) to Eqs.(10) and (11), we calculate 2 dimensional nonlinear simultaneous equation with respect to w and t. Then, we use the Newton type homotopy equation:

$$\begin{array}{l} h(w,t) = F(w,t) - (1-t)F(w_0, t_0), \\ F(w,t) = z_v w + z_e - 2 \cdot z(w), \\ z(w) = (\log f)_{xx}(w) \end{array} \tag{12}$$

In Eq.(12), we substitute $x = wx_v - xe$, $y = wy_v - x_e$ into $\frac{\partial^2}{\partial x^2}(\log f)$ and define $z(w)$. The starting point in homotopy method is set as $(w_0, t_0) = (0, 0)$, and let $a_{1,0} = 0.01$, $b_0 = 0.03$ and $n_1 = 3$.

Figs. 3 and 4 show images of visualization of the 2-soliton at $k_i = 1.0$ and $k_2 = 1.5$. The number of pixel, the viewpoint and the center of screen for each image are given in Table 1. It is noted that the viewpoint of Figure 4 is closer to the 2-soliton surface than the one of Figure 3. From Figs.3 and 4, we can see that the collision of 2 isolated pulses is visualized by the proposed method.

The previous work[6][7] was able to show that the method using hyper-ellipsoid was more effective than the method using hyper-sphere. Therefore, the method which achieved the best results in previous work is compared with the proposed method.Table 2 shows the computational time by both of the proposed method and the conventional one under the Ultra SPARC IIi 334MHz. From table 2, the proposed method can reduce computational cost.

Table 1: the parameters of Fig.1 and Fig.2

	Fig.3	Fig.4
number of pixels	150×157	150×157
viewpoint	$(-41, -14, 24)$	$(1, -5, 1.5)$
center of screen	$(-5, -1, -20)$	$(-2, -2, -18.5)$

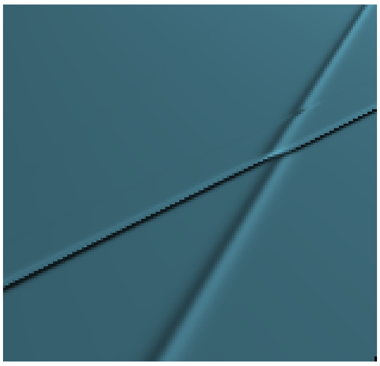

Figure 3: an image of 2 soliton pulse (1)

Figure 4: an image of 2 soliton pulse (2)

Table 2: the computational time

	Fig.3 [s]	Fig.4 [s]
The conventional method	31.040	36.540
The proposed method	23.930	28.440

4. Conclusion

In this paper, we proposed an improved solution path tracing algorithm based on hyper-elliposoid, and compare with the conventional one by applying to the ray tracing method to visualize a certain type of free form surfaces. In the proposed method, width of the defined ellipsoid with respect to each axis is variable according to the width of the domain and/or the nonlinearity. In the case of visualization of a surface by the ray tracing, the width of domain (parameters for the surface and the length of the ray) is usually different very much. Therefore, the proposed method is suitable to the case. In the case of 2 soliton pulse's image, the proposed method is more effective than the conventional one. In the future, it is necessary to apply to more complicated nonlinear equations and to confirm validity.

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Keywords homotopy method, path tracing, nonlinear equation, hyper-sphere, hyper-ellipsoids, computer graphics(CG)

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