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Solving a Kind of Optimization Problems by Using SPICE Program

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Abstract

Analog Hopfield neural networks (HNNs) have so far been used to solve many kinds of optimization problems, in particular, combinatorial problems such as a TSP. When we solve a minimization problem with equality constraints by using HNNs, however, the constraints are satisfied only approximately.

In this paper we propose a method to satisfy the equality condition accurately. The equality constraints are realized via Kirchhoff's voltage law (KVL) in some loops which contain several ideal transformers and dc sources. We analyze the above circuit by using the SPICE program. We can show that the dynamic behavior of the circuit just corresponds to the Lagrange multiplier method.

We applied the proposed method to several kinds of optimization problems, such as the Linear Programming, Geometric Programming, and Quadratic Programming. Many computer simulations show that our method works very well.

Keywords: optimization problem, equality constraints, Lagrange multiplier method.

1. Introduction

The optimization problem discussed here is as follows:
Minimize the objective function:

$$g_0(x) = \sum_{i=0}^{l} f_i(x) \tag{1}$$

subject to:

$$g_k(x) = \sum_{j=1}^{n} a_{kj} x_j + a_{k0} = 0 \quad (k = 1, 2, \cdots, m) \tag{2}$$

$$x_j \geq 0 \quad (j = 1, 2, \cdots, n) \tag{3}$$

Here, $x = [x_1, x_2, \cdots, x_n]^T$, $g_0(x)$ is a nonlinear function of x, and m is the number of constraints.

The variables x_j are associated with the values of voltages across capacitors or currents through inductors. For simplifying the explanation, the variables discussed here are related to capacitor voltages. The variables x_j attaining the minimum value of the objective function can be obtained as a steady state dc voltages across the capacitors, in the circuit where the derivative of the objective function equals zero and the constraints are satisfied.

The circuit we proposed consists roughly of two parts. One part is composed of sub-circuits realizing the derivative of the objective function and the other part sub-circuits realizing the equality constraints. The former sub-circuits consist of direct current sources, voltage-controlled voltage sources (VCVS), voltage-controlled current sources (VCCS), operational amplifiers and so on. The latter sub-circuits consist of some ideal transformers and dc voltage sources. Besides, in order to ensure the variables $x_j \geq 0$, some ideal diodes must be connected in parallel with the corresponding capacitors.

2. Sub-circuits

The sub-circuits for realizing the derivative of the objective function will be shown for some examples in Sec.4. In this section we mainly describe a sub-circuit for realizing the equality constraints in Eq.(2). The circuit configuration is shown in Fig.1 where the turn ratios of the ideal transformers correspond to the coefficients a_{kj} in Eq.(2). We can see that KVL of each loop in the left hand side of the ideal transformers is given as follows:

$$E_k = \sum_{j=0}^{n} a_{kj} x_j \tag{4}$$

Therefore the constraints in Eq.(2) can be exactly satisfied by setting $E_k = -a_{k0}$.

On the other side, we can see from Fig.1 that there are two currents, which are related to the current through the capacitor C_j. One flows left ward of the capacitor

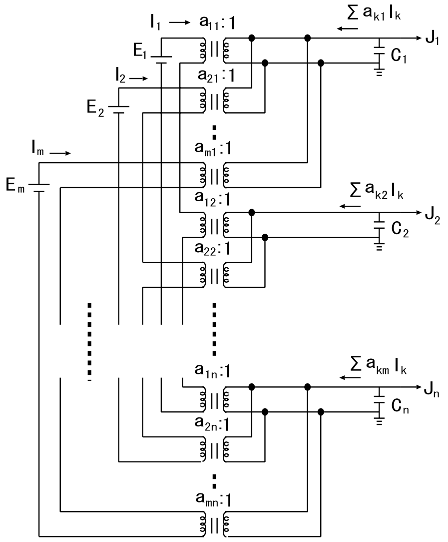

Figure 1: The sub-circuit for realizing the equality constraints

C_j that can be written by

$$\sum_{k=1}^{m} a_{kj} I_k \tag{5}$$

The other denoted by J_j flows right ward and is controlled by the sub-circuits for realizing the derivative of the objective function. In fact, we just design the sub-circuits for J_j to be satisfied with the equation as follows:

$$J_j = \frac{\partial g_0(x)}{\partial x_j} \tag{6}$$

Therefore, based on Kirchhoff's current law (KCL), the following circuit equation can be obtained easily.

$$C_j \frac{dx_j}{dt} + J_j + \sum_{k=1}^{m} a_{kj} I_k = 0 \tag{7}$$

3. Optimization

The dynamic behavior of our circuit just corresponds to the Lagrange multiplier method. Here, the constraints are limited to linear constraints as in Eq.(2). The optimization problem is equivalent to minimization of the function as follows:

$$\tilde{g}_0(x) \equiv g_0(x) + \sum_{k=1}^{m} \lambda_k (\sum_{j=1}^{n} a_{kj} x_j + a_{k0}) \tag{8}$$

where λ_k are Lagrange coefficients with no restrictions. So what we should do is to get the value of the variables x^* when

$$\frac{\partial \tilde{g}_0(x^*)}{\partial x_j} = 0 \tag{9}$$

From Eq.(8), we know that

$$\frac{\partial \tilde{g}_0(x)}{\partial x_j} = \frac{\partial g_0(x)}{\partial x_j} + \sum_{k=1}^{m} \lambda_k a_{kj} \tag{10}$$

Comparing Eq.(10) with Eqs.(6) and (7), we can define that

$$\lambda_k \equiv I_k \tag{11}$$

Thus,

$$C_j \frac{dx_j}{dt} = -\frac{\partial \tilde{g}_0(x)}{\partial x_j} \tag{12}$$

From Eq.(12), we can see that when the circuit reaches the steady state, the voltages across the capacitors are constants and the left hand side of Eq.(12) becomes zero, so the derivative of the function $\tilde{g}_0(x)$ will be zero too. Therefore, the objective function is minimized.

We define the function $\tilde{g}_0(x)$ as the energy function and the following inequality is satisfied.

$$\frac{d\tilde{g}_0(x)}{dt} \leq 0 \tag{13}$$

It means the network is stable. The proof of Ineq.(13) is omitted here.

4. Simulation

In this section, we give some examples to illustrate the effectiveness of the proposed method in solving this kind of constrained optimization problem. The simulation is carried out by means of the SPICE program and its results show that the presented circuitis are always globally stable at a equilibrium point.

Example 1 : Let us consider the following quadratic objective function with linear inequality constraints:

Minimize:

$$f(x) = x_1^2 + 2x_2^2 + x_3^2 - x_1x_2 + x_2x_3 + 3x_1 + 2x_2 + 0.25x_3 \tag{14}$$

subject to:

$$\begin{cases} 2x_1 + x_2 + x_3 \geq 6 \\ x_1 + 2x_2 + x_3 \leq 8 \end{cases} \tag{15}$$

This problem has a unique optimal solution $x^* = [2.096, 0.538, 1.269]^T$. The derivative of $f(x)$ can be written as

$$\frac{\partial f(x)}{\partial x} = \begin{pmatrix} 2x_1 - x_2 + 3 \\ -x_1 + 4x_2 + x_3 + 2 \\ x_2 + 2x_3 + 0.25 \end{pmatrix} \tag{16}$$

We use the circuit in Fig.2 to solve the above problem and the results are shown in Fig.3.

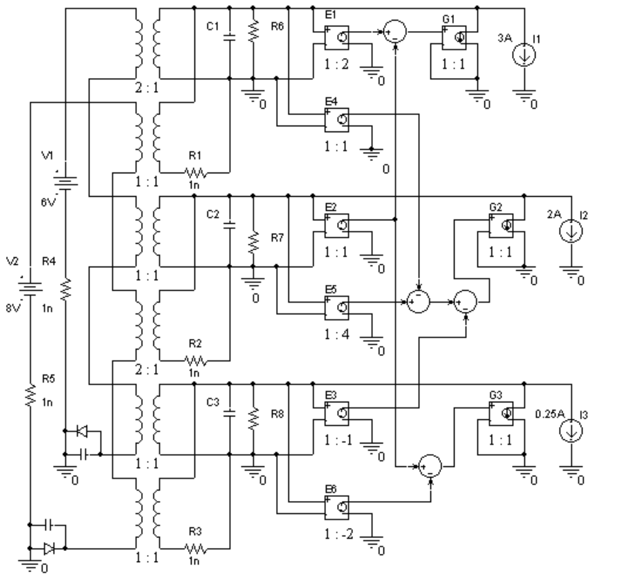

Figure 2: The circuit for solving Example 1.

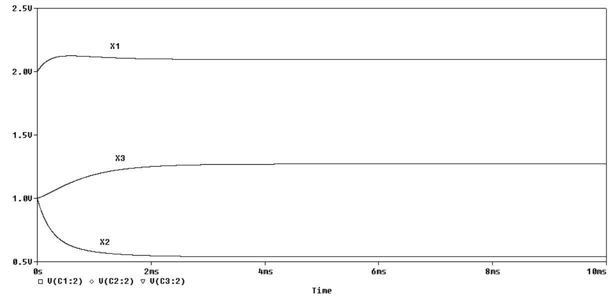

Figure 3: The transient behavior of the circuit in Fig.2.

Example 2 : Let us consider the following nonlinear convex optimization problem with bound constraints:
Minimize:

$$\begin{aligned} f(x) = 11x_1^2 + 10(x_2-2)^2 + (x_3-2x_4)^2 \\ +20x_1x_2 - 40x_1 + e^{x_1-x_2+x_3+x_4} \end{aligned} \tag{17}$$

subject to:

$$d \le x \le h \tag{18}$$

where $d = [0,1,0,0]^T$ and $h = [1,2,1,1]^T$. This problem has a unique optimal solution $x^* = [0,2,0,0]^T$. The derivative of $f(x)$ can be calculated as

$$\frac{\partial f(x)}{\partial x} = \begin{pmatrix} 2x_1 + 20(x_1+x_2-2) + e^{x_1-x_2+x_3+x_4} \\ 20(x_1+x_2-2) - e^{x_1-x_2+x_3+x_4} \\ e^{x_1-x_2+x_3+x_4} + 2(x_3-2x_4) \\ e^{x_1-x_2+x_3+x_4} - 4(x_3-2x_4) \end{pmatrix} \tag{19}$$

We use the circuit in Fig.4 to solve the above problem and the results are shown in Fig.5.

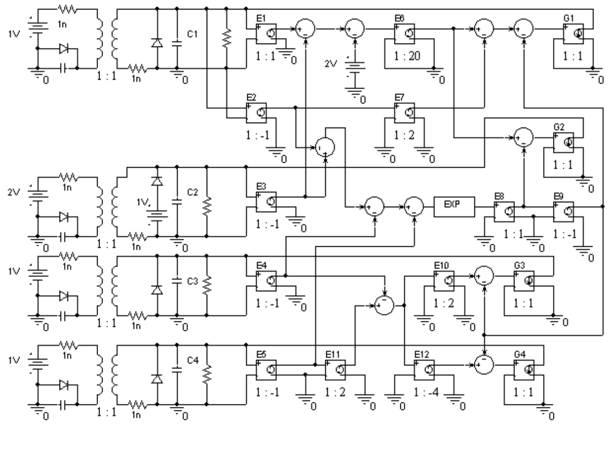

Figure 4: The circuit for solving Example 2.

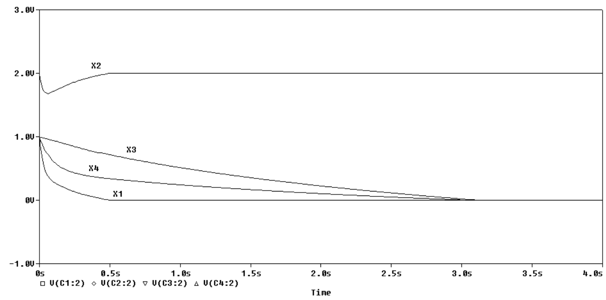

Figure 5: The transient behavior of the circuit in Fig.4.

Example 3 : Let us consider the following constrained geometric-programming problem.
Minimize:

$$f(x) = 3x_2x_3 + \frac{1}{9}x_1 \tag{20}$$

subject to:

$$\frac{1}{3}x_1^{\frac{1}{2}}x_2^{-2}x_3^{\frac{1}{2}} + 3x_1^{-1}x_2x_3^{-2} \le 1 \tag{21}$$

The dual problem can be stated as follows:
Maximize:

$$g(\lambda) = \left(\frac{3}{\lambda_1}\right)^{\lambda_1}\left(\frac{\frac{1}{9}}{\lambda_2}\right)^{\lambda_2}\left(\frac{\frac{1}{3}}{\lambda_3}\right)^{\lambda_3}\left(\frac{3}{\lambda_4}\right)^{\lambda_4}(\lambda_3+\lambda_4)^{\lambda_3+\lambda_4} \tag{22}$$

subject to:

$$\begin{cases} \lambda_1 + \lambda_2 = 1 \\ \lambda_2 + \frac{1}{2}\lambda_3 - \lambda_4 = 0 \\ \lambda_1 - 2\lambda_3 + \lambda_4 = 0 \\ \lambda_1 + \frac{1}{2}\lambda_3 - \lambda_4 = 0 \end{cases} \tag{23}$$

Thus the nonlinear constraint converts into linear constraints. For simplification, we can change the objective

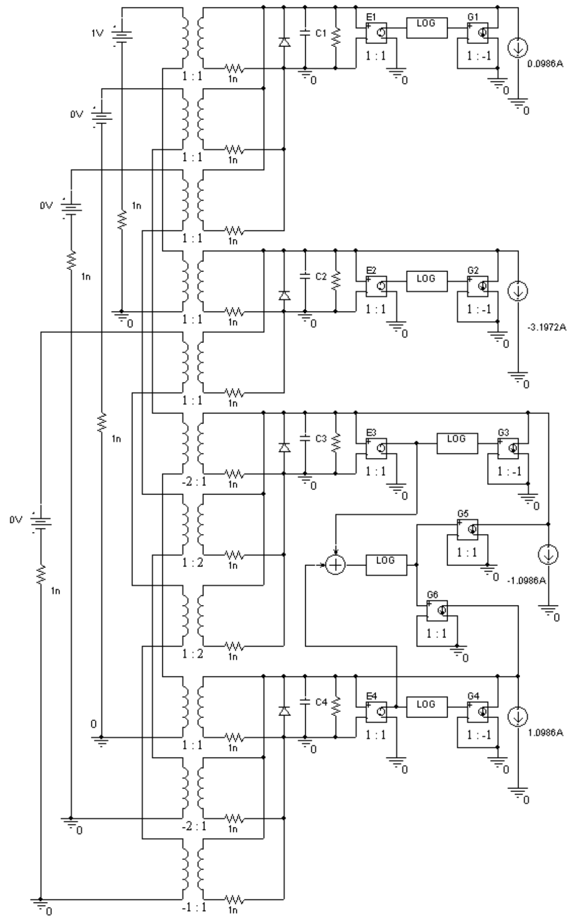

Figure 6: The circuit for solving Example 3.

function into the function as follows:

$$\ln g(\lambda) = \lambda_1 \ln\left(\frac{3}{\lambda_1}\right) + \lambda_2 \ln\left(\frac{\frac{1}{9}}{\lambda_2}\right) + \lambda_3 \ln\left(\frac{\frac{1}{3}}{\lambda_3}\right) + \lambda_4 \ln\left(\frac{3}{\lambda_4}\right) + (\lambda_3+\lambda_4)\ln(\lambda_3+\lambda_4) \tag{24}$$

The problem has a unique optimal solution $\lambda^* = [\frac{7}{9}, \frac{2}{9}, \frac{2}{3}, \frac{5}{9}]^T$. The derivative of the above function can be written as

$$\frac{\partial \ln g(\lambda)}{\partial \lambda} = \begin{pmatrix} -\ln\lambda_1 + 0.0986 \\ -\ln\lambda_2 - 3.1972 \\ -\ln\lambda_3 + \ln(\lambda_3+\lambda_4) - 1.0986 \\ -\ln\lambda_4 + \ln(\lambda_3+\lambda_4) + 1.0986 \end{pmatrix} \tag{25}$$

We use the cirucit in Fig.6 to solve the above problem and the results are shown in Fig.7. The optimal primal objective function value is

$$\left(\frac{27}{7}\right)^{\frac{7}{9}} \left(\frac{1}{2}\right)^{\frac{2}{9}} \left(\frac{1}{2}\right)^{\frac{2}{3}} \left(\frac{27}{5}\right)^{\frac{5}{9}} \left(\frac{11}{9}\right)^{\frac{11}{9}} \approx 5.03274$$

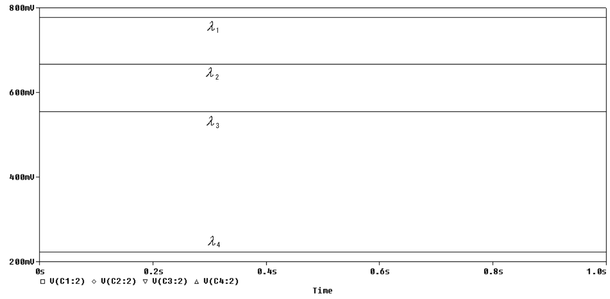

Figure 7: The transient behavior of the circuit in Fig.6.

5. Conclusion

In this paper, we have presented a method for solving a kind of optimization problems. In contrast to conventional HNNs, the constraints can be satisfied exactly and the exact optimal value can be obtained. Furthermore, the stability of the presented circuit is very well, and the speed of convergence is very fast. The proposed circuit can be applied to solve various linear constrained optimization problems including Linear Programming, Quadratic Programming, Nonlinear Programming such as Geometric Programming and so on. Simulation results show that the proposed circuit is effective in solving these optimization problems.

Further investigations will be aimed at the extension of the constraints and let the circuit be applied to solve a more general projection problem.

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