Identification of the Jump in Switched Linear Systems

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1. Introduction

Hybrid systems have attracted many researchers especially in control and computer communities[1]. Although the identification of hybrid systems becomes more important, there are few researches of identification of hybrid systems.

In general, we must specify the following four items in order to identify a hybrid system:

(1) continuous dynamics in each location,

(2) *jump* of the continuous state at the transition of the discrete state,

(3) switching conditions, or *guard*,

(4) a location of discrete state of each time instance.

The item (1) may be obtained by an ordinary least squares method if the continuous dynamics is represented by a linear system. One of the problems peculiar to the identification of hybrid systems is (2). If we restrict the jump of the continuous state to an affine transformation, the jump is represented by a matrix depending on the coordinate systems. This means that even if there is no jump in a certain coordinate, there might be a jump in another coordinate. The identification of the local continuous dynamics will be done in a particular coordinate in general, there probably exists jumps of the continous state at the discrete transitions. The problems of (3) and (4) are also important in the identification of hybrid systems and will be formulated by a clustering[2]. However, we will not treat these problems here, *i.e.*, we assume we know the time sequence of the discrete transitions and we know which location each interval of the input-output data belongs to.

In this paper, we address the problem of identifying discrete-time piece-wise affine system, in which the jump of the continuous state at the discrete transition is represented by an affine transformation. We assume that the input-output data is already classified to locations. We also assume that the duration time in each location is longer than the degree of the local dynamic systems. Under these conditions, we derive an ARX model with jumps in the observer canonical form. And we show we can identify each local linear systems and the transition matrix of each jump by using least squares method.

2. Preliminaries

Definition 1 *Hybrid automaton[1]: A hybrid automaton is described by a septuple* $(L, X, A, W, E, Inv, Act)$ *where the symbols have the following meanings.*

L is a set of discrete states. or locations, $X \subset R^n$ is a set of continuous state space.

A is a finite set of symbols.

$W = R^q$ is a continuous commutication space.

E is a finite set of edges called transitions or events. Every edge is defined by a five-tuple $(l, a, Guard_{ll'}, Jump_{ll'}, l')$, where $l, l' \in L$, $a \in A$, $Guard_{ll'} \subset X$, and $Jump_{ll'} \subset X \times X$. The transition from the discrete state l to l' is enabled when $x \in Guard_{ll'}$, while x jumps to a value x' given by $(x, x') \in Jump_{ll'}$.

Inv is a mapping from the locations L to the set of subsets of X.

Act is a mapping that assigns to each location $l \in L$ a continuous dynamics.

3. Problem Statement

Consider a discrete time piecewise affine system described by a hybrid automaton of Definition 1, in which Act is described by

$$x(k+1) = A_l x(k) + b_l u(k) + F_l \tag{1}$$

$$y(k) = c_l x(k) \tag{2}$$

and $Jump_{ll'}$ and $Guard_{ll'}$ in E is described by

$$Jump_{ll'} = \{(x, x') \in X \times X | x' = T_{ll'} x + s_{ll'}\} \tag{3}$$

$$Guard_{ll'} = \left\{ x \in X | C_{ll'} \begin{pmatrix} x \\ 1 \end{pmatrix} = 0 \right\} \tag{4}$$

4. Parametlized Plant

Assume each local dynamics is observable. Then equation (1) can be realized by an oberver canonical form. Without loss of generality, A_l, b_l, c_l and F_l have the following form.

$$A_l = \begin{pmatrix} -a_{l,1} & 1 & & & \\ -a_{l,2} & & 1 & & \\ \vdots & & & \ddots & \\ \vdots & & & & 1 \\ -a_{l,n} & & & & 0 \end{pmatrix}, \tag{5}$$

$$b_l = \begin{pmatrix} b_{l,1} \\ \vdots \\ \vdots \\ b_{l,n} \end{pmatrix}, \qquad F_l = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f_l \end{pmatrix} \tag{6}$$

$$c_l = (\, 1 \quad 0 \quad 0 \quad \ldots \quad 0 \,)\,. \tag{7}$$

The state vector x can be expressed as a linear combination of $X(k) = (y(k-1), y(k-2), \ldots, y(k-n), u(k-1), u(k-2), \ldots, u(k-n))^{\mathrm{T}}$ and f_l as:

$$x(k) = (\, \mathrm{Hankel}(-a_l) \quad \mathrm{Hankel}(b_l) \,)\, X(k) + \bar{f}_l \tag{8}$$

where $\bar{f}_l = (f_l, \ldots, f_l)^{\mathrm{T}}$ and $\mathrm{Hankel}(\cdot) : R^n \to R^{n \times n}$ is defined as

$$\mathrm{Hankel}(x) = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ x_2 & & \iddots & 0 \\ \vdots & \iddots & & \vdots \\ x_n & 0 & \ldots & 0 \end{pmatrix}. \tag{9}$$

Assume that a transition from l to l' occurs at time $k = \tau$, and that $x(\tau_-)$ jumps to $x(\tau_+)$ as $x(\tau_+) = T_{ll'} x(\tau_-) + s_{ll'}$, then $X(\tau_-)$ must jump to $X(\tau_+)$ such that

$$\begin{aligned} &(\, \mathrm{Hankel}(-a_{l'}) \quad \mathrm{Hankel}(b_{l'}) \,)\, X(\tau_+) \\ &= T_{ll'} \,(\, \mathrm{Hankel}(-a_l) \quad \mathrm{Hankel}(b_l) \,)\, X(\tau_-) \\ &\quad + T_{ll'} \bar{f}_l + s_{ll'} - \bar{f}_{l'}. \end{aligned} \tag{10}$$

Let τ_i be a switching time, and the discrete state lies in a location $P_i \in L$ for $k \in [\tau_i, \tau_{i+1})$. Then, at time $k \in [\tau_i + n, \tau_{i+1})$, the system must be described by a usual ARX model:

$$\begin{aligned} y(k) =& -a_{P_i,1} y(k-1) - \cdots - a_{P_i,n} y(k-n) \\ & + b_{P_i,1} u(k-1) + \cdots + b_{P_i,n} u(k-n) + f_{P_i}. \end{aligned} \tag{11}$$

On the other hand, at time $k \in [\tau_i, \tau_i + n - 1)$, the system is influenced by the jump of the continous state, it must be described by the following equations:

$$\begin{aligned} Y(\tau_i) =& (\, \mathrm{Toep}(-a_{P_i}) \quad \mathrm{Toep}(b_{P_i}) \,) \begin{pmatrix} Y(\tau_i) \\ U(\tau_i) \end{pmatrix} + \bar{f}_{P_i} \\ & + (\, \mathrm{Hankel}(-a_{P_i}) \quad \mathrm{Hankel}(b_{P_i}) \,)\, X(\tau_{i+}) \end{aligned} \tag{12}$$

$$\begin{aligned} =& (\, \mathrm{Toep}(-a_{P_i}) \quad \mathrm{Toep}(b_{P_i}) \,) \begin{pmatrix} Y(\tau_i) \\ U(\tau_i) \end{pmatrix} \\ & + T_{P_{i-1}P_i} \,(\, \mathrm{Hankel}(-a_{P_{i-1}}) \quad \mathrm{Hankel}(b_{P_{i-1}}) \,)\, X(\tau_i) \\ & + T_{P_{i-1}P_i} \bar{f}_{P_{i-1}} + s_{P_{i-1}P_i} \end{aligned} \tag{13}$$

where

$$\begin{aligned} Y(\tau) &= (y(\tau), y(\tau+1), \ldots, y(\tau+n-1))^{\mathrm{T}}, \\ U(\tau) &= (u(\tau), u(\tau+1), \ldots, u(\tau+n-1))^{\mathrm{T}}, \end{aligned}$$

and $\mathrm{Toep}(\cdot) : R^n \to R^{n \times n}$ is defined as

$$\mathrm{Toep}(x) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ x_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ x_{n-1} & \cdots & x_1 & 0 \end{pmatrix}. \tag{14}$$

Note that $X(\tau_i)$ includes only the observed data, while $X(\tau_{i+})$ does not.

5. Estimation Algorithm

Assume that it is given a time sequence of the input and the output of the hybrid system $u(1), \ldots, u(N)$ and $y(1), \ldots, y(N)$, that there are K times of jumps and the switching times are given as $\tau_1, \ldots, \tau_K$, and that the location $P_k \in L$ of each time is given.

In the plant model described in the previous section, the parameters to be identified are (a_l, b_l, f_l) for $l \in L$ and $T_{ll'}$, $s_{ll'}$ for $l, l' \in L$. Because there are product forms of $T_{ll'}$ and (a_l, b_l, f_l) in (13), we will identify a parameter matrix $P_{ll'}$

$$P_{ll'} = (\, T_{ll'} \,(\, \mathrm{Hankel}(-a_l) \quad \mathrm{Hankel}(b_l) \,) \quad T_{ll'} \bar{f}_l + s_{ll'} \,)\,.$$

Thus, the paramter vector to be identified is given by

$$\begin{aligned} \theta =& (-a_1^{\mathrm{T}}, b_1^{\mathrm{T}}, f_1, \ldots, -a_L^{\mathrm{T}}, b_L^{\mathrm{T}}, f_L, \\ & p_{111}, \ldots, p_{11(2n+1)}, \ldots, \\ & p_{LL1}, \ldots, p_{LL(2n+1)})^{\mathrm{T}}. \end{aligned} \tag{15}$$

where $p_{ll'i}$ is the i-th row of $P_{ll'}$. The matrix Φ, which is formed by the regressor vectors and is associated with the parameter vector θ and the output data vector

$$Y = (y(n), y(n+1), \ldots, y(N))^{\mathrm{T}}, \tag{16}$$

can be constructed as a $(2K+1)\times L(1+L)$ block matrix as follows:

$$(2i-1, P_i)\text{-block of } \Phi{=}\Phi_i \tag{17}$$
$$(2i, P_{i+1})\text{-block of } \Phi{=}\Phi^{(1)}_{i+1,i} \tag{18}$$
$$(2i, LP_i+P_{i+1})\text{-block of } \Phi{=}\Phi^{(2)}_{i+1,i} \tag{19}$$
$$\text{other block of } \Phi{=}0 \tag{20}$$

where $l, l' \in L$ and Φ_i, $\Phi^{(1)}_{ij}$, $\Phi^{(2)}_{ij}$ are the matrices defined as:

$$\Phi_i{=}\left(\begin{pmatrix} X(\tau_i+n) \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} X(\tau_{i+1}-1) \\ 1 \end{pmatrix}\right)^{\mathrm{T}} \tag{21}$$
$$\Phi^{(1)}_{i,i-1}{=}\begin{pmatrix} \mathrm{Toep}(Y(\tau_i)) & \mathrm{Toep}(U(\tau_i)) & \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \end{pmatrix} \tag{22}$$
$$\Phi^{(2)}_{i,i-1}{=}I_n \otimes \begin{pmatrix} X^{\mathrm{T}}(\tau_i) & 1 \end{pmatrix}. \tag{23}$$

By using a least squares method[3], the estimated paramters will be obtained as

$$\hat{\theta} = (\Phi^{\mathrm{T}}\Phi)^{-1}\Phi^{\mathrm{T}}Y. \tag{24}$$

6. Numerical Example: Bouncing Ball

In order to illustrate how the proposed method works, numerical simulations of bouncing ball is presented.

We assume the mass of the ball is 1kg, the viscosity is 0.001, the ball bounces at the height 1m with the reflection rate 0.8, *i.e.* the dynamicics is represented as

$$\ddot{y}(t) + 0.001\dot{y}(t) = u(t) - 9.8 \tag{25}$$

while the transition rule is

$$\dot{y}(\tau_+) = -0.8 \times \dot{y}(\tau_-) \quad \text{if} \quad y(\tau) = 1.0. \tag{26}$$

The control input $u(t)$ is defined so that the ball continues to bounce in addition to an M-sequence signals. The input/output data are sampled for each 1/64[sec] (Fig.1) and there are quantization errors to 10 bit data. The data length is 1024 and there are 29 times of jumps.

In the following, some subscript will be omitted since the set of locations $L = \{1\}$. The discrete system equivalent to the continous system (25) is:

$$x(k+1){=}Ax(k) + bu(k) - \begin{pmatrix} 0 \\ f \end{pmatrix} \tag{27}$$
$$y(k){=}\begin{pmatrix} 1 & 0 \end{pmatrix} x(k) \tag{28}$$

while the transition rule becomes

$$x(\tau_+) = Tx(\tau) + s \quad \text{if} \quad Cx(\tau) = 1 \tag{29}$$

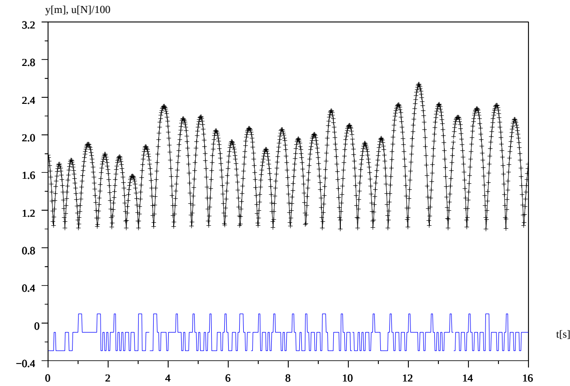

Figure 1: Data set of the bouncing ball system

where

$$A{=}\begin{pmatrix} 1.9999844 & 1 \\ -0.9999844 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0.0001221 \\ 0.0001221 \end{pmatrix},$$
$$f{=}0.0023926, \quad C = \begin{pmatrix} 1.00 & 0 \end{pmatrix},$$
$$T{=}\begin{pmatrix} -0.8 & 0 \\ 0 & -0.8 \end{pmatrix}, \quad s = \begin{pmatrix} 1.8 \\ -1.7999719 \end{pmatrix}.$$

The identified results are following:

$$\hat{A}{=}\begin{pmatrix} 1.9990816 & 1 \\ -0.9994356 & 0 \end{pmatrix}, \quad \hat{b} = \begin{pmatrix} 0.0001968 \\ 0.0000260 \end{pmatrix},$$
$$\hat{f}{=}0.0021093, \quad \hat{C} = \begin{pmatrix} 1.0396368 & 0.0403059 \end{pmatrix},$$
$$\hat{T}{=}\begin{pmatrix} -0.7920729 & -0.0050663 \\ 0.0672236 & -0.7898083 \end{pmatrix}, \quad \hat{s} = \begin{pmatrix} 1.8052873 \\ -1.8718076 \end{pmatrix}.$$

Fig.2 shows the simulation result based on the identified parameters, the initial state is estimated by an observer and the data of $t \in [0, 2]$. Because the system is not asymptotically stable, the error is accumulated and does not converge to 0. However, in the first 2 bouncings, the curve seems to fit well to the original data. So, the obtained model is considered to be sufficient for synthesis of an observer.

$Guard = \{x \in X | Cx = 1\}$ may be estimated by clustering the set of $\hat{x}(\tau_{i-})$'s and the set of $\hat{x}(\tau_i - 1)$'s where τ_i is a switching time. The result is shown in Fig.3. Because the data is almost reduced to a one dimendional space near the switching surface, we must abondon the precise estimation of C.

In the proposed method, the matrix Φ is too large that its condition number might be very large. In fact, the condition number of Φ in this numerical example becomes

$$\mathrm{cond}(\Phi) = 20153.06$$

The matrix used in the estimation of T and s from $(\hat{a}, \hat{b}, \hat{f})$ and $\hat{P}$ has its condition number 5.8291656. To-

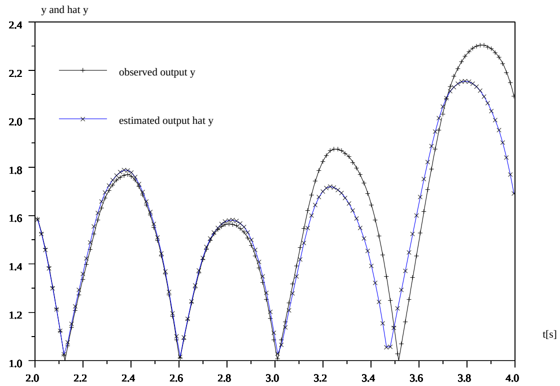

Figure 2: Observed and estimated output based on the proposed method

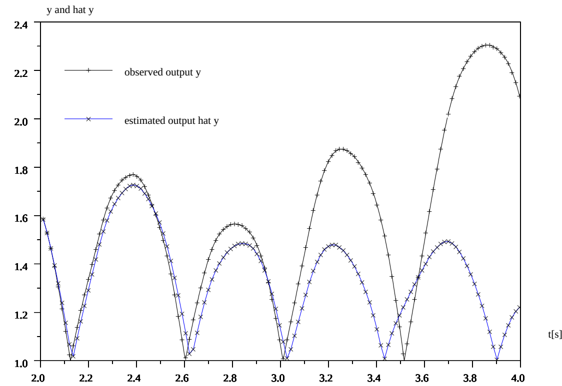

Figure 4: Observed and estimated output based on the subdivided method

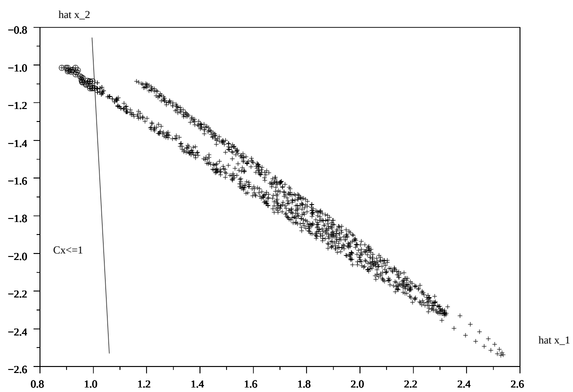

Figure 3: Estimation of the guard $\{x|Cx \leq 1\}$

tally, the condition number of the whole estimation becomes $20153.06 \times 5.8291656 = 117475.52$. It is very large from a numerical point of view.

In order to reduce the condition number, the parameter estimation problem is subdivided into two parts: namely, an estimation of $(\hat{a}, \hat{b}, \hat{f})$ and an estimation of $(\hat{T}, \hat{s})$. The condition number of the matrix in the estimation of $(\hat{a}, \hat{b}, \hat{f})$ is 650.24451, while The condition number of the matrix in the estimation of $(\hat{T}, \hat{s})$ is 188.9771. The total condition number will be $650.24451 \times 188.9771 = 122881.32$. Totally, one can not expect a better estimation, one might obtain a better estimation of $(\hat{a}, \hat{b}, \hat{f})$. The results are following:

$$\hat{A}=\begin{pmatrix} 1.9990599 & 1 \\ -0.9994190 & 0 \end{pmatrix}, \quad \hat{b} = \begin{pmatrix} 0.0001999 \\ -0.0000233 \end{pmatrix},$$

$$\hat{f}=0.0020901, \quad \hat{C} = \begin{pmatrix} 1.0225384 & 0.0241474 \end{pmatrix},$$

$$\hat{T}=\begin{pmatrix} -0.7842900 & 0.0131603 \\ 0.0427005 & -0.8254826 \end{pmatrix}, \quad \hat{s} = \begin{pmatrix} 1.798452 \\ -1.8666466 \end{pmatrix}.$$

There are few improvements in numbers. Fig.4 shows the time response of the estimated system. The error of the resonse becomes worse than the proposed method. A reason why the subdivision of the problem does not lead to an improvement might be the sparseness and the special block structure of the matrix Φ in the proposed method.

7. Conclusion

An identification algorithm is proposed to estimate a piecewise affine system with affine jumps. Thought the condition nubmer tends to become large, the proposed method has an advantage that it can be extended to an on-line algorithm. In order to illustrate how the proposed method works, it is shown a numerical simulation of a bouncing ball. The bouncing ball system is not asymptotically stable that the estimation error results in a large output errors. However, the estimated model is well applicable for the synthesis of an observer.

References

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