Nonlinear Frequency-coupled Analysis in the Correlated Multiplicative and Additive Noises

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Abstract-In this paper, the nonlinear frequency-coupled harmonics analysis in the correlated multiplicative and additive noise are studied. First, the cross-mixing concept is proposed to characterize the approximate independence not only in the time domain but also among the random processes. Then, several properties on the mixing concept are given. These properties show that the cross-mixing assumption is weaker than the assumption that self-mixing and independence assumption . Finally ,we prove that the proposed algorithms based on the time-averaged moment polyspectral can be used to detect the nonlinear coupled-frequency with the cross-mixing assumption and these properties. The simulation shows that the algorithm in this paper is effective.

1. Introduction

When the input of a nonlinear system is harmonics signal , the output of the nonlinear system has not only the input harmonic frequency, but also the coupled frequencies . This is phenomena can be used to detect and analyze the nonlinearity of the system. This nonlinear frequency-coupled harmonics analysis has wide applications in oceanography ,seismology ,EEG analysis ,manufacturing ,laser velocimetry , and plasma physics applications . Some researchers study the nonlinear frequency-coupled in the additive noise. In recent years, Zhou[1] research the nonlinear frequency-coupled harmonics in the multiplicative and additive noise with the conditions that the multiplicative and additive noise are independent and satisfy the self-mixing assumption , which describes the approximate independence of the random process in the time domain. In fact , it is difficult to guarantee that the multiplicative and additive noises are independent. Therefore, it is important to study the problem without the prior information on whether the background noises are independent.

The structure of this paper is as follows. In section 2, the cross-mixing concept is proposed to depict the random processes' approximate independence both in the time domain and in process set. The Properties on the cross-mixing and the relationship between cross-mixing and mixing concepts are presented combined with the measurement model. An approach to frequency-coupled analysis in the correlated multiplicative and additive noise is presented in section 3. In section 4, Simulations are shown. In section 5, We conclude the paper with some remarks.

2. Cross-mixing and self-mixing

2.1 The definition of cross-mixing and self-mixing

First, we present the definition of the cross-cumulant and self-cumulant. The vector $S(t)=[s_0(t),s_1(t),\cdots,s_L(t)]^T$ is a statistic vector. The (k+1)th-order cross-cumulant of the vector $S(t)$ is

$$C^{(P)}_{s_{l_1},s_{l_2},\cdots,s_{l_{(k+1)}}}(t;\tau_1,\tau_2,\cdots,\tau_k)\overset{\Delta}{=}cum\{s^{(p_1)}_{l_1}(t),s^{(p_2)}_{l_2}(t+\tau_1)$$
$$\cdots,s^{(p_{(k+1)})}_{l_{(k+1)}}(t+\tau_k)\},l_1,\cdots,l_{(k+1)}=0,\cdots,L \qquad (1)$$

where P is (k+1)-D column vector which element is +1 or –1. We make the notation $s^{(p_1)}_{l_1}(t)$ to identify $s_{l_1}(t)$ if $p_1=1$ and $s^*_{l_1}(t)$ if $p_1=-1$ .When $s_{l_1}(t)=s(t),s_{l_2}(t)=s(t),\cdots,s_{l_{(k+1)}}(t)=s(t)$,we obtain the expression of the self-cumulant of the s(t) statistic process as follow.

$$C^{(P)}_{(k+1)s}(t;\tau_1,\tau_2,\cdots,\tau_k)=cum\{s^{(p_1)}(t),s^{(p_2)}(t+\tau_1),$$
$$\cdots,s^{(p_{(k+1)})}(t+\tau_k)\} \qquad (2)$$

If $s_l(t)$'s are mutually independent, the cross-cumulant of the vector is equal to zero. Also, if $s(t)$ is independent in time , the self-cumulant of the statistic process $s(t)$ is equal to zero. That is to say, we can use the self-cumulant to describe the independence degree of one statistic process in time .

Definition 1 [2]: statistic process $s(t)$ is mixing if the cumulant of $s(t)$ is absolutely summable .

$$\sum_{\boldsymbol{\tau}}\left|C^{(\mathbf{P})}_{(k+1)s}(t;\boldsymbol{\tau})\right|<\infty,\ \boldsymbol{\tau}=[\tau_1,\cdots,\ \tau_k]\ ,\forall k \qquad (3)$$

where $C^{(\mathbf{P})}_{(k+1)s}(t;\boldsymbol{\tau})=C^{(P)}_{(k+1)s}(t;\tau_1,\tau_2,\cdots,\tau_k)$

In fact, if s(t) satisfy the mixing condition(3), the process s(t) is approximately independent in time. The linear and nonlinear processes with stable kernels are all mixing. In order to study harmonic retrieval and frequency-coupled analysis in dependent multiplicative and additive noise , we need to characterize the approximate independence of the correlated multiplicative and additive noise in the time domain and in process set if processes are looked as a set.

Definition 2: statistic processes vector $S(t)$ are cross-mixing if the cross-cumulant of $s_l(t)^{'}s$ are absolutely summable.

$$\sum_{\boldsymbol{\tau}}\left|C^{(\mathbf{P})}_{s_{l_1},s_{l_2},\cdots,s_{l(k+1)}}(t;\boldsymbol{\tau})\right|<\infty\ ,\forall k \qquad (4)$$

where

$C^{(\mathbf{P})}_{s_{l_1},s_{l_2},\cdots,s_{l(k+1)}}(t;\boldsymbol{\tau})=C^{(P)}_{s_{l_1},s_{l_2},\cdots,s_{l(k+1)}}(t;\tau_1,\tau_2,\cdots,\tau_k)$.

Contrast to the cross-mixing , the mixing can be called self-mixing. The self-mixing concept describes that the process is well separated in time domain. While the cross-mixing concept depict that the processes are well separated not only in time domain but also in process set.

2.2 properties of the cross-mixing and self-mixing

We will first give the observed model.

$$x(t)=\sum_{l=1}^{L}s_l(t)e^{j\omega_l t}+s_0(t)\ ,t=0,1,\cdots,T-1 \qquad (5)$$

a1) $\omega_l^{'}s$ is distinct in $(-\pi,0)\cup(0,\pi]$.

a2) $s_l(t)^{'}s$ are multiplicative noise, $l=1,\cdots,L$, $s_0(t)$ are additive noise. They are all stationary statistic processes.

Prosperity 1 All elements of the discrete statistic process set are cross-mixing if every element of the set $\{s_l(t)\}_{l=0}^{L}$ satisfies the self-mixing condition(3) and is independent of each other.

Proof:

Due to the independence of the element, we infer

$$C^{(\mathbf{P})}_{s_{l_1},s_{l_2},\cdots,s_{l(k+1)}}(t;\boldsymbol{\tau})\overset{\Delta}{=}cum\{s^{(p_1)}_{l_1}(t),s^{(p_2)}_{l_2}(t+\tau_1),\cdots$$

$$,s^{(p_{(k+1)})}_{l_{(k+1)}}(t+\tau_k)\}\delta(l_1,l_2,\cdots,l_{(k+1)}) \qquad (6)$$

where $\delta(.)$ is defined as follow.

$$\delta(l_1,l_2,\cdots,l_{(k+1)})=\begin{cases}1\ ,\ l_1=l_2=\cdots=l_{(k+1)}\\ 0\ ,\ \ otherwise\end{cases} \qquad (7)$$

The absolute sum of the cross-cumulant is as follow,

$$\sum_{\boldsymbol{\tau}}\left|C^{(\mathbf{P})}_{s_{l_1},s_{l_2},\cdots,s_{l(k+1)}}(t;\boldsymbol{\tau})\right|=\sum_{\boldsymbol{\tau}}\left|cum\{s^{(p_1)}_{l_1}(t),s^{(p_2)}_{l_2}(t+\tau_1),\cdots\right.$$

$$\left.\cdots,s^{(p_{(k+1)})}_{l_{(k+1)}}(t+\tau_k)\}\right|\delta(l_1,l_2,\cdots,l_{(k+1)})$$

$$\le\sum_{\boldsymbol{\tau}}\left|cum\{s^{(p_1)}_{l_1}(t),s^{(p_2)}_{l_1}(t+\tau_1),\cdots,s^{(p_{(k+1)})}_{l_1}(t+\tau_k)\}\right|$$

(8)

Because every element of set is self-mixing, we can infer that the processes of the set are cross-mixing.

$$\sum_{\boldsymbol{\tau}}\left|C^{(\mathbf{P})}_{s_{l_1},s_{l_2},\cdots,s_{l(k+1)}}(t;\boldsymbol{\tau})\right|\le\infty \qquad (9)$$

The two assumptions that the multiplicative and additive noise are self-mixing and independent of each other are made in many papers[1,3] on the harmonic retrieval and frequency-coupled analysis. In fact, the self-mixing assumption is to describe the one element‘s approximate independence in time. And the independence assumption is to depict the relationship of each other. From this property , we can see that the cross-mixing assumption is weaker than the self-mixing and independence of each other assumption. It can also be seen that the cross-mixing can depict the approximate independence in time domain and in process set simultaneously.

Property 2 **T**he outputs $\{s_l(t)\}_{l=0}^{L}$ of the stable systems with impulse response function $\{h_l(t)\}_{l=0}^{L}$ satisfy the cross-mixing condition(4) the input process $e(n)$,if the input process e(n),
which self-cumulant exists, is stationary and white noise with zero mean.

$$\sum_{\boldsymbol{\tau}}\left|C^{(\mathbf{P})}_{s_{l_1},s_{l_2},\cdots,s_{l(k+1)}}(\boldsymbol{\tau})\right|<\infty\ ,\forall k \qquad (10)$$

From property 2,we conclude that many linear processes are cross-mixing.

Property 3 The observed signal $x(t)$ is self-mixing if the multiplicative and additive noise are cross-mixing.

From property 3 and property 1, we conclude that the results ,which can be achieved with the assumption that the multiplicative and additive noise are independent of each other and self-mixing, can be also obtained with the assumption that the multiplicative and additive noise are cross-mixing. In other words, the cross-mixing assumption is weaker than the assumptions of self-mixing and independence in time domain and in processes set. Property is useful to analyze the asymptotic performance of the time-averaged moment polyspectrum.

3. Frequency-coupled analysis

The previous paper[1] had studied the frequency-coupled analysis with the assumptions of self-mixing and independence in time domain and in processes set. In this section, we will obtain common results on the frequency-coupled in the dependent multiplicative and additive noise. We give the additional assumptions,

A3) $\omega_l^{'}s$ satisfy the inequality $\omega_m + \omega_n \neq 0, m, n = 1, \cdots, L$.

A4) The processes set $\{s_l(t)\}_{l=0}^{L}$ composed of the multiplicative noise with nonzero mean and additive noise with zero mean are cross-mixing.

The observed signal is a cyclostationary process. The time-varying high-order moment of the observed process is defined as

$$m_{(k+1)x}(t;\tau) \stackrel{\Delta}{=} E\{x^*(t)x(t+\tau_1)\cdots x(t+\tau_k)\} \quad (11)$$

And the time-averaged moment is defined as

$$\overline{m}_{(k+1)x}(\tau) \stackrel{\Delta}{=} \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} m_{(k+1)x}(t;\tau) \quad (12)$$

The time averaged moment polyspectrum, which full name is generalized FS time-averaged moment polyspectrum, is also defined as

$$\overline{M}_{(k+1)x}(\lambda) \stackrel{\Delta}{=} \lim_{T\to\infty} \frac{1}{T^k} \sum_{\tau=0}^{T-1} \overline{m}_{(k+1)x}(\tau) e^{-j\lambda\tau^{\mathrm{T}}} \quad (13)$$

where λ is denoted as $\lambda = [\lambda_1, \lambda_2, \cdots, \lambda_k]$ .An estimator of time-averaged moment polyspectrum is defined as

$$\hat{M}_{(k+1)x}(\lambda) = \frac{1}{T^{(k+1)}} X_T(\lambda_1)\cdots X_T(\lambda_k) X^*{}_T(\lambda_1 + \cdots \lambda_k) \quad (14)$$

where $X_T(\lambda_1)$ is the DFT transform of the observed signal $x(t)$ .It can be simply inferred that (14) is an asymptotical unbiased estimator of the FS time-averaged moment polyspectrum.

Theorem :let us consider the third-order time-averaged moment polyspectrum defined as (14). If $x(t)$ is given by (5) and satisfy A1)~A4) assumptions, then

the time-averaged moment polyspectrum is given as follow:

$$\overline{M}_{3x}(\lambda) = \sum_{l_1=1}^{L}\sum_{l_2=1}^{L}\sum_{l_3=1}^{L} m_{l_1}^* m_{l_2} m_{l_3} \delta(\omega_{l_1} - \omega_{l_2} - \omega_{l_3}) \delta(\lambda_1 - \omega_{l_2})\delta(\lambda_2 - \omega_{l_3}) \quad (15)$$

where $m_l \stackrel{\Delta}{=} E\{s_l(t)\}$ and $\delta(\lambda)$ is kronecker function defined as

$$\delta(\lambda) = \begin{cases} 1, & \lambda = 0 \bmod 2\pi \\ 0, & otherwise \end{cases}$$

Proof is given in appendix. We should point out that the result in [1]is obtained with the independence in processes set and self-mixing assumptions. The theorem 1 in present paper is achieved with the cross-mixing assumption. We have inferred that the cross-mixing assumption is weaker than the self-mixing and independence in time domain assumptions . From theorem 1, we can find that the third-order time-averaged moment polyspectrum is not equal to zero only if there exists the nonlinear frequency-coupled phenomena .

4. Simulation Results

In simulation , the measurement signal ,which data length is 1024, comprises four harmonic signal with $\omega_1 = 1.0, \omega_2 = 1.5, \quad \omega_3 = 2.5, \omega_4 = 1.9$. The multiplicative and additive noise are generated by the following systems:

$$S_1(t) = e(i) + 0.3e(i-1) + 0.2e(i-2)$$
$$S_2(t) = e(t) + 0.4e(t-1) + 0.3e(t-2)$$
$$S_3(t) = e(t) + 0.25e(t-1) + 0.15e(t-2)$$
$$S_4(t) = e(t) + 0.55e(t-1) + 0.45e(t-2)$$
$$v(t) = e(t) + 0.15e(t-1) + 0.25e(t-2)$$

where $e(t)$ is i.i.d complex Gaussian noise with $E\{e(t)\} = 1$ and variance $\sigma_e^2 = 1$. The 3-dimension plot and contour plot of the third-order time-averaged moment spectral are given in Fig.1 and Fig.2 .

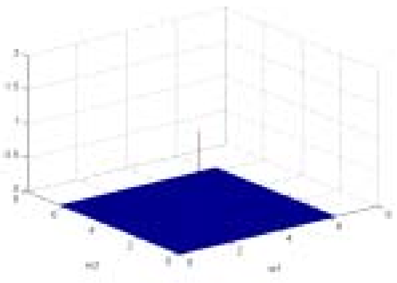

Fig1 Third-order time-averaged moment polyspectral

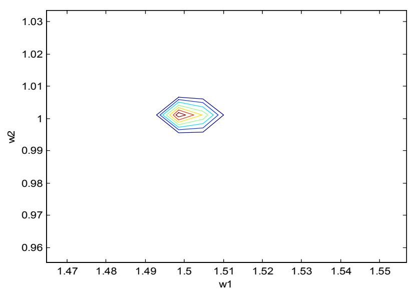

Fig2 The contour plot of Third-order time-averaged moment polyspectral

The peak of 3-dimesion shows that there exists the nonlinear frequency-coupled phenomenon. And contour plot shows peak position at (1.5,1).

5. Conclusion

In this paper, the cross-mixing concept is proposed to characterize the approximate independence not only in the time domain but also among the random processes. Then, several properties on the mixing concept are given. These properties show that the cross-mixing assumption is weaker than the assumption that self-mixing and independence assumption . Finally ,we prove that the proposed algorithms based on the time-averaged moment polyspectral can be used to detect the nonlinear coupled-frequency harmonics with the cross-mixing assumption and these properties.

Appendix

Proof :

let us consider the time-averaged moment of the observed process.

$$\overline{m}_{3x}(\tau) = \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} m_{3x}(t;\tau)$$

$$= \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} E\{x^*(t)x(t+\tau_1)x(t+\tau_2)\} \qquad (16)$$

From assumption A3) and the definition of the kronecker function , we write time-averaged moment as

$$\overline{m}_{3x}(\tau) = \sum_{l_1=1}^{L}\sum_{l_2=1}^{L}\sum_{l_3=1}^{L} E\{S_{l_1}^*(t)S_{l_2}(t+\tau_1)S_{l_3}(t+\tau_2)\}$$

$$\times\delta(\omega_{l_1}-\omega_{l_2}-\omega_{l_3})e^{j\omega_{l_2}\tau_1}e^{j\omega_{l_3}\tau_2} + \sum_{l_1=1}^{L}\sum_{l_3=1}^{L} E\{S^*{}_{l_1}(t)$$

$$S_0(t+\tau_1)S_{l_3}(t+\tau_2)\}\delta(\omega_{l_1}-\omega_{l_3})e^{j\omega_{l_3}\tau_2} + \sum_{l_1=1}^{L}\sum_{l_2=1}^{L}$$

$$E\{S^*{}_{l_1}(t)(S_{l_2}(t+\tau_1)S_0(t+\tau_2)\}\delta(\omega_{l_1}-\omega_{l_2})e^{j\omega_{l_2}\tau_1}$$

$$+ E\{S_0(t)S_0(t+\tau_1)S_0(t+\tau_2)\} \qquad (17)$$

The second and third term of (17)s not equal to zero Only if $l_1 = l_3$, $l_1 = l_2$. we obtain

$$\overline{m}_{3x}(\tau) = \sum_{l_1=1}^{L}\sum_{l_2=1}^{L}\sum_{l_3=1}^{L} E\{S_{l_1}^*(t)S_{l_2}(t+\tau_1)S_{l_3}(t+\tau_2)\}$$

$$\times\delta(\omega_{l_1}-\omega_{l_2}-\omega_{l_3})e^{j\omega_{l_2}\tau_1}e^{j\omega_{l_3}\tau_2} \qquad (18)$$

$$+\sum_{l_1=1}^{L} E\{S^*{}_{l_1}(t)S_0(t+\tau_1)S_{l_1}(t+\tau_2)\}e^{j\omega_{l_1}\tau_2} \qquad (19)$$

$$+\sum_{l_1=1}^{L} E\{S^*{}_{l_1}(t)S_{l_1}(t+\tau_1)S_0(t+\tau_2)\}e^{j\omega_{l_1}\tau_1} \qquad (20)$$

$$+ E\{S_0(t)S_0(t+\tau_1)S_0(t+\tau_2)\} \qquad (21)$$

Applying the cumulant-to-moment equation to (18)we obtain

$$E\{S_{l_1}^*(t)S_{l_2}(t+\tau_1)S_{l_3}(t+\tau_2)\} =$$

$$cum\{S_{l_1}^*(t), S_{l_2}(t+\tau_1), S_{l_3}(t+\tau_2)\} \qquad (22)$$

$$+ cum\{S_{l_1}^*(t)\}cum\{S_{l_2}(t+\tau_1), S_{l_3}(t+\tau_2))\} \qquad (23)$$

$$+ cum\{S_{l_2}(t+\tau_1)\}cum\{S_{l_1}^*(t), S_{l_3}(t+\tau_2)\} \qquad (24)$$

$$+ cum\{S_{l_3}(t+\tau_2)\}cum\{S_{l_1}^*(t), S_{l_2}(t+\tau_1)\} \qquad (25)$$

$$+ cum\{S_{l_1}^*(t)\}cum\{S_{l_2}(t+\tau_1)\}cum\{S_{l_3}(t+\tau_2)\} \qquad (26)$$

Because the noises satisfy the cross-mixing condition , (22) has no contribution to time-averaged moment polyspectrum. Let us consider (23)'s contribution to time-averaged moment polyspectrum. If we define $\tau = \tau_2 - \tau_1$, then we obtain

$$\left|\lim_{T\to\infty}\frac{1}{T^2}\sum_{\tau_1}\sum_{\tau_2} cum\{S_{l_1}^*(t)\}cum\{S_{l_2}(t+\tau_1), S_{l_3}(t+\tau_2)\}\}\right|$$

$$\le \left|m_{l_1}\right|\lim_{T\to\infty}\frac{1}{T}\sum_{\tau=-(T-1)}^{T-1}\left|C_{S_{l_2}S_{l_3}}(\tau)\right|\left|(1-\frac{|\tau|}{T})\right|$$

$$\le \left|m_{l_1}\right|\lim_{T\to\infty}\frac{1}{T}\sum_{\tau=-(T-1)}^{T-1}\left|C_{S_{l_2}S_{l_3}}(\tau)\right| = 0 \qquad (27)$$

Therefore the (23) has no contribution to the time-averaged moment polyspetrum. We can also verify that (24) and (25) have no contribution to the time-averaged moment polyspectrum. Only (26)'s time-averaged moment polyspectrum is not equal to zero. If we apply the cumulant-to-moment equation to (19),(20) and (21), we can also prove that they have no contribution to time-averaged moment polyspectrum.

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