Precise Control of Chaos by Using a Modified OGY Method

Toshiaki Higa, Yoshinori Egami, Wang Zheng, and Kenji Murao

Miyazaki University
1-1 Gakuen Kibanadai Nishi, Miyazaki, Japan
Phone:+81-985-58-7403, Fax:+81-985-58-7403
Email: murao@esl.miyazaki-u.ac.jp

Abstract

A simple modified-OGY method is presented to control chaos by which the controlled trajectory can automatically converge to the unknown true target and control signal becomes almost zero after having controlled chaos.

1. Introduction

In 1990 Ott, Grebogi, and Yorke proposed a novel method of controlling chaos[1]. The method, now called the OGY method, stabilizes one of the many unstable periodic orbits embedded in a chaotic attractor, through only small time-dependent perturbations in such a way that when an iterate falls near the desired orbit, the next iterate will be forced back toward the stable manifold of the original orbit. The OGY method has attracted the attention of many researchers interested in applications of nonlinear dynamics in various fields. Many variants of this method have appeared in the literature[2].

However, when we apply the OGY method to control chaos using an estimated fixed-point (or a periodic-orbit) which has inevitable errors caused by approximation and noise, there are problems such that the controlled fixed-point might be slightly different from the true one and a small amount of control signal must be kept applying on the system. In 1996 Xu and Bishop proposed a method for self-locating control of chaotic systems using Newton algorithm[2]. This method seems to be a general and systematic approach for the above-mentioned problems. We also have conducted research on these problems independently[4] apart from the method of Xu and Bishop. Our method is based on the observation of graphical properties of the OGY method and simple, although at present it is not as general as the Xu and Bishop's method

In this paper, we show a simple modified-OGY method to control chaos by which the chaotic system can automatically converge to the unknown true target and the control signal becomes very small after having controlled chaos. To improve the estimations of the fixed-point we use a Newton-like method which is very different from the one Xu and Bishop employed. The Radial Basis Function (RBF) with parsimonious error criterion[3] is also utilized to obtain a good initial estimation of the fixed point and the Jacobian from time-series data with observation noise.

2. The OGY method

The method of OGY[1] is briefly summerized in this section. This method is based on the observation that unstable periodic orbits are dense in a chaotic attractor. The method assumes that the dynamics of the system can be represented by a k-dimensional nonlinear map (e.g., by a surface of section or time one return map)

$$\boldsymbol{x}(t+1) = \boldsymbol{f}(\boldsymbol{x}(t), p) \qquad \boldsymbol{x}(t) \in \boldsymbol{R}^k \tag{1}$$

where $\boldsymbol{x}(t)$ is the state of the system at a discrete time t and p is some accessible system parameter. We suppose that the parameter p can be varied in a small range about some nominal value p^* but that the local dynamics about it do not vary much with the small change in p. Let $\boldsymbol{x}_{F(p^*)}$ be the unstable fixed point for p^* on the attractor which one wants to stabilize. We change p slightly from p^* to p'. Then the fixed point will shift to a new position $\boldsymbol{x}_{F(p')}$. For small perturbation $p' - p^*$, we approximate

$$\boldsymbol{g} \equiv \frac{\partial \boldsymbol{f}}{\partial p}\Big|_{\boldsymbol{x}_{F(p^*)}} \approx \frac{\boldsymbol{x}_{F(p')} - \boldsymbol{x}_{F(p^*)}}{p' - p^*} \tag{2}$$

, which allows an experimental determination of the vector $\boldsymbol{g}$. We assume $\boldsymbol{g}$ does not vary much with the small change in p. Near the fixed point $\boldsymbol{x}_{F(p)}$, we can use a linear approximation

$$\boldsymbol{x}(t+1) - \boldsymbol{x}_{F(p)} \approx \boldsymbol{J}(\boldsymbol{x}(t) - \boldsymbol{x}_{F(p)}) \tag{3}$$

where $\boldsymbol{J}$ is the Jacobian matrix $\boldsymbol{J} \equiv \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\Big|_{\boldsymbol{x}_{F(p^*)}}$. We assumed $\boldsymbol{J}$ does not vary much with the small change around p^*. Let $\delta\boldsymbol{x}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{F(p^*)}, \delta p = p - p^*$. From Eq.(2) and its assumption we can write $\boldsymbol{x}_{F(p)} \approx \boldsymbol{x}_{F(p^*)} + \delta p\boldsymbol{g}$. Using these relations we can recast Eq.(3)

$$\delta\boldsymbol{x}(t+1) - \delta p\boldsymbol{g} \approx \boldsymbol{J}(\delta\boldsymbol{x}(t) - \delta p\boldsymbol{g}) \ . \tag{4}$$

To simplify explanation we consider two-dimensional discrete dynamical systems. In this case

$$\boldsymbol{J} = \lambda_u \boldsymbol{e}_u \boldsymbol{v}_u + \lambda_s \boldsymbol{e}_s \boldsymbol{v}_s \tag{5}$$

with $\boldsymbol{e}_u(\boldsymbol{e}_s)$ the unstable(stable) eigendirections of $\boldsymbol{J}$ with eigenvalues $\lambda_u(\lambda_s)$ and $\boldsymbol{v}_u(\boldsymbol{v}_s)$ their contravariant basis vectors, i.e.,

$$\boldsymbol{v_u e_u} = \boldsymbol{v_s e_s} = 1, \quad \boldsymbol{v_u e_s} = \boldsymbol{v_s e_u} = 0 \ . \tag{6}$$

The condition that $\boldsymbol{x}(t+1)$ falls on the local stable manifold of the fixed point $\boldsymbol{x}_{F(p^*)}$ can be formulated as $\boldsymbol{v}_u \delta \boldsymbol{x}(t+1) = 0$, which yields the control formula[1] for the new value of the control parameter $p = p^* + \delta p(t)$,

$$\delta p(t) = \frac{\lambda_u}{\lambda_u - 1} \frac{\boldsymbol{v}_u}{\boldsymbol{v}_u \boldsymbol{g}} \delta \boldsymbol{x}(t) \ . \tag{7}$$

The parameter perturbations are activated only if $\delta \boldsymbol{x}(t)$ falls in sufficiently small neighborhood of the fixed point and $\delta p(t)$ is less than the maximal allowed disturbance δp_{max}; otherwise $\delta p(t)$ is set to 0.

3. Radial basis functions (RBF)

We use the parsimonious RBF method presented by Mees[3]. We simply call this method the parsimonious RBF. We are given a set of pairs $\{(\boldsymbol{y}_i, \boldsymbol{x}_i)\}$ $(i = 1, ..., T)$ where it is assumed that

$$\boldsymbol{y}_i = \boldsymbol{f}(\boldsymbol{x}_i), \ \boldsymbol{x}_i \in \boldsymbol{R}^k, \ \boldsymbol{y}_i \in \boldsymbol{R}^k, \ (i = 1, \cdots, T) \ . \tag{8}$$

Consider how to find approximation $\hat{\boldsymbol{f}} = \{\hat{f}_l(\boldsymbol{x})\}$ to $\boldsymbol{f}$. The parsimonious RBF uses the ARB(affine plus radial basis) model

$$\hat{f}_l(\boldsymbol{x}) = \boldsymbol{\alpha}_l \cdot \boldsymbol{x} + \beta_l + \sum_{j=1}^{N} \lambda_{lj} \phi(|\boldsymbol{x} - \boldsymbol{c}_j|) \tag{9}$$

with an affine term $\boldsymbol{\alpha}_l \cdot \boldsymbol{x} + \beta_l \ \ (l = 1, \cdots, k)$, a given radial basis function ϕ and given centers of the RBF $\{\boldsymbol{c}_j\}$, $j = 1, \ldots, N$. In this paper we use the cubic function $\phi(r) = r^3$ for the radial basis function.

To determine a good model size (the number of parameters in Eq.(9)), the parsimonious RBF uses the Schwarz Criterion

$$\mathrm{SIC}(m) = T \log\left(\frac{\boldsymbol{\varepsilon}_m^2}{T}\right) + m \log(T) \tag{10}$$

where m denotes the mth stage of the orthogonalization process, $\boldsymbol{\varepsilon}_m$ the error vector at this stage and T the number of data. The number m (denoted by M) that gives the minimum in the SIC(m) curve should define a good model size and gives the stopping condition in the orthogonalization process.

4. An algorithm for automatic correction

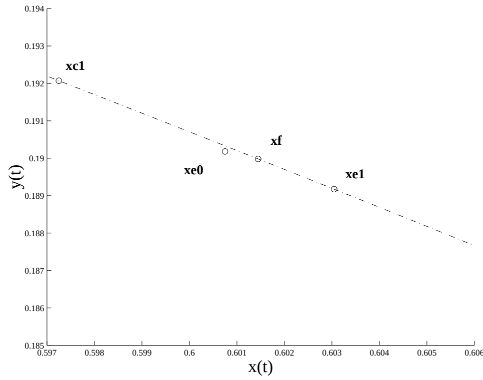

Fig. 1-a Iterated estimations of the fixed point for Ikeda map

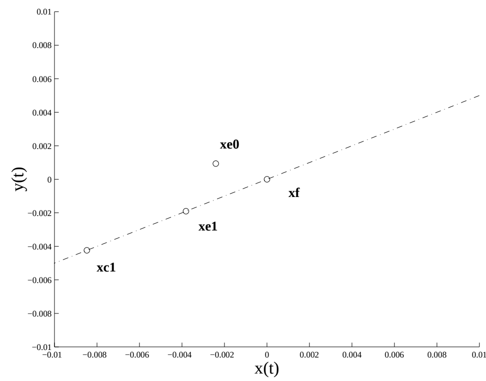

Fig. 1-b Iterated estimations of the fixed point for Z-type Map

The conventional OGY method uses an estimated fixed-point for a control paramater value p^* without improving the estimation. In the proposed method the estimation of the fixed-point is automatically improved during the OGY controlling process. A prototype solution for this problem was given in [4]. In this paper we develop a method to accelerate the convergence to the exact fixed-point by using a Newton-like method.

We assume that the local dynamics about the fixed-point do not vary much with the allowed small changes in control parameter p as in the conventional OGY. We employ the parsimonious RBF to construct k-dimensional nonlinear map $\boldsymbol{x}_{n+1} = \boldsymbol{f}(\boldsymbol{x}_n, p)$ using observed data from the chaotic system. Our method is based on the facts that were observed in numerical experiments of the conventional OGY method. These facts are in the followings:

(1) The initially estimated fixed-point $\boldsymbol{x}_{e0}$ is slightly different from the true one $\boldsymbol{x}_f$ as shown in Figure 1-a and Figure 1-b. Using $\boldsymbol{x}_{e0}$ as a control target the OGY stabilize the system at a point $\boldsymbol{x}_{c0}$ that becomes a new estimation $\boldsymbol{x}_{e1}$ for the fixed-point. Observe that the point $\boldsymbol{x}_{e1}$ is on the line through $\boldsymbol{x}_f$

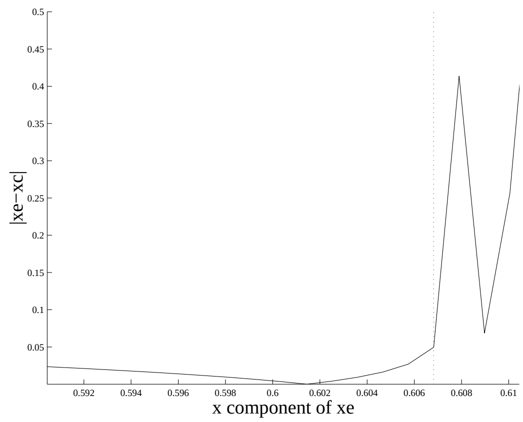

Fig. 2-a Characteristic of $|\boldsymbol{x}_e - \boldsymbol{x}_c|$ vs. x-component of $\boldsymbol{x}_e$ for Ikeda map

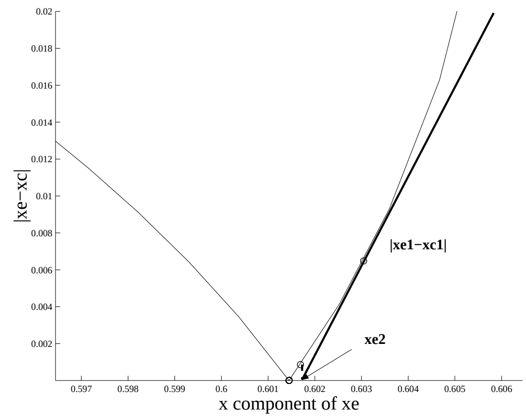

Fig. 3-a Characteristic of $|\boldsymbol{x}_e - \boldsymbol{x}_c|$ vs. x-component of $\boldsymbol{x}_e$ closed up near the fixed point for Ikeda map

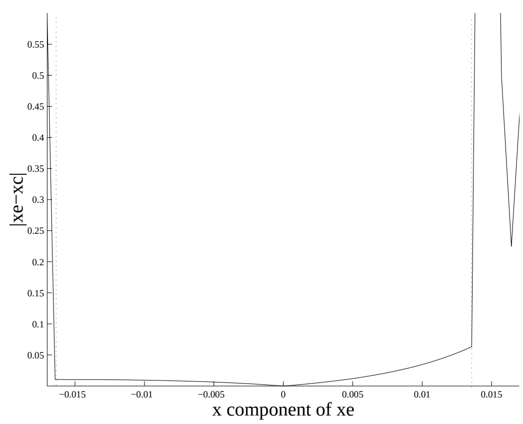

Fig. 2-b Characteristic of $|\boldsymbol{x}_e - \boldsymbol{x}_c|$ vs. x-component of $\boldsymbol{x}_e$ for Z-type map

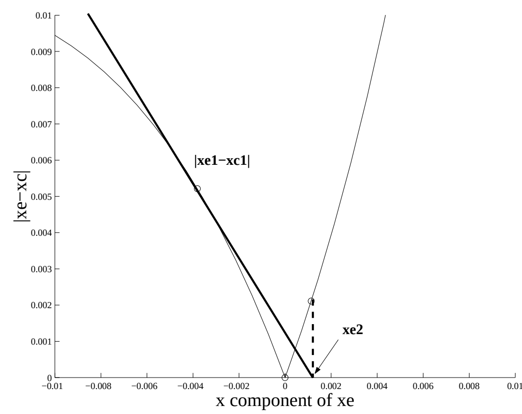

Fig. 3-b Characteristic of $|\boldsymbol{x}_e - \boldsymbol{x}_c|$ vs. x-component of $\boldsymbol{x}_e$ closed up near the fixed point for Z-type map

(see the Figures) and the slope of the line is given by the slope vector $\boldsymbol{g}$ in [1]. Observations show that iterated estimations $\boldsymbol{x}_{ei}$ are all on the line but not converge to the true $\boldsymbol{x}_f$.

(2) Near the true fixed point, plotting the difference between $\boldsymbol{x}_e$ and its corresponding $\boldsymbol{x}_c$ versus x-component of $\boldsymbol{x}_e$ shows a smooth curve near the fixed-point as shown in Figure 2-a and Figure 2-b. Close-up of these figures near the fixed-point is shown in Figure 3-a and Figure 3-b (thin curves). Note that the "zero" point of the curve is the true fixed-point.

From the above observations, it seems that we can precisely estimate the true fixed point. However, we don't know the precise function of the curve beforehand. Hence we approximate the curve by straight line determined by two points at each iterate: the point ($|\boldsymbol{x}_{ei} - \boldsymbol{x}_{ci}|$, x-component of $\boldsymbol{x}_{ei}$) and a close corresponding point on the curve (see thick lines in Figure 3-a and Figure 3-b). The intersection of the line with the abscissa gives a new approximation of $\boldsymbol{x}_f$. This iterated improvement is almost the same with the Newton method that has the second order convergence rate, and very simple.

5. Numerical examples

Numerical experiments are performed to verify the effectiveness of the proposed method. We consider the following two examples which are 2-dimensional discrete maps with accessible control parameter perturbation p:
Example A: Ikeda Map

$$\begin{aligned} x(t+1) &= a(t) + b(x(t)\cos(d) - y(t)\sin(d)) \\ y(t+1) &= b(x(t)\sin(d) + y(t)\cos(d)) \end{aligned}$$

where

$$d = 0.4 - \frac{6}{(1 + x^2(t) + y^2(t))}$$

$$a(t) = a_0 + p(t),\ a_0 = 1.0,\ b = 0.7.$$

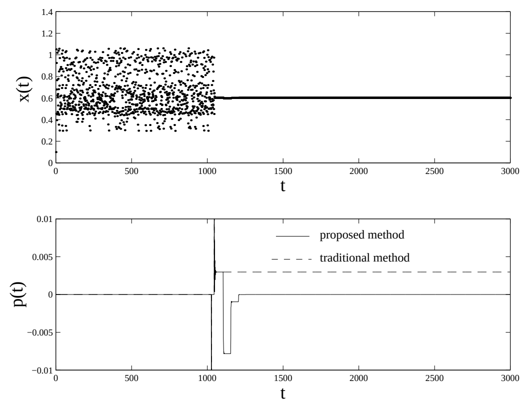

Fig. 4-a A time series of $x(t)$ and the time history of control-parameter perturbation $p(t)$ for Ikeda map

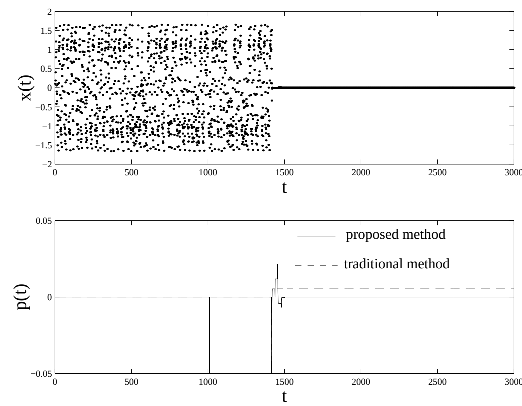

Fig. 4-b A time series of $x(t)$ and the time history of control-parameter perturbation $p(t)$ for Z-type Map

Table 1-a Errors of controlled fixed-points and perturbations for Ikeda map with noise

noise	0.5%	1.0%
conventional method's $\boldsymbol{x}_f$	7.2655e-004	1.4686e-003
proposed method's $\boldsymbol{x}_f$	5.7150e-012	2.6865e-012
conventional method's $p(t)$	2.9752e-003	6.9085e-002
proposed method's $p(t)$	-6.3600e-013	4.7398e-012

Table 1-b Errors of controlled fixed-points and perturbations for Z-type map with noise

noise	0.5%	1.0%
conventional method's $\boldsymbol{x}_f$	5.3352e-003	7.5311e-003
proposed method's $\boldsymbol{x}_f$	4.8558e-013	5.7710e-013
conventional method's $p(t)$	5.3352e-003	8.1278e-003
proposed method's $p(t)$	2.6426e-014	-7.5797e-013

Example B: Z-type Map

$$\begin{aligned} x(t+1) &= ax(t) - x(t)^3 + y(t) + c(t) \\ y(t+1) &= bx(t) \end{aligned}$$

with $c(t) = p^* + p(t),\ p^* = 0,\ a = 1.9,\ b = 0.5$.

As a preparatory step for controlling chaos, we reconstruct 2-dimensional map $\hat{\boldsymbol{f}}$ from observed time series data $\{x(t), y(t)\}$ using the parsimonious RBF, and find an initial estimation of the aimed fixed-point, the Jacobian $\boldsymbol{J}$ and gradient $\boldsymbol{g}$ around the fixed-point. Next we apply the OGY method to control chaos using updated estimations of the fixed-point presented in Section 4.

Figure 4-a and Figure 4-b show time series of $x(t)$ and the time history of control-parameter perturbation $p(t)$ for both Ikeda map and Z-type map respectively. Note that the present method gives very small perturbation after having controlled while the conventional OGY keeps applying perturbation even after having control.

Table 1-a and Table 1-b show the control of chaos generated by Ikeda map and Z-type map respectively with additional observation noise. Note that the proposed method can control chaos into the true target very precisely within the error order 10^{-12} and control parameter input $p(t)$ is very small (within the order 10^{-13}) after completion of control.

6. Conclusions

We have presented a simple modified-OGY method to control chaos. The method enables chaotic systems to automatically converge to the exact fixed point at the given control-parameter p^*. Numerical experiments show that the proposed method is very effective for 2-dimensional discrete maps. Extension of this method to higher-dimensional case and plural control parameters case remains as future works.

References

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