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Fuzzy Metrics for Risk Analysis of Uncertain Systems

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1. Introduction

In general, many systems are designed and controlled by human, so that some kinds of fuzziness are introduced into the behavior of the systems. The systems with the fuzziness is called uncertain systems. Recently because the uncertain systems become more complex and large scale, it is very difficult to analyze the behavior or risk of the systems. Here the risk means danger and that exists in many phases of uncertain systems, for example, the design, the behavior, the maintenance of the systems, et al. In spite of increase of gravity to discuss the risk, there is few concrete method to analyze the risk.

One of main purpose of our works is to develop the methodology to analyze risk of such uncertain systems from the global standpoint. In such methodology, the concept of fuzzy metric plays very important role. The concept of fuzzy sets was proposed in Ref.[1], which is one of most important tools to analyze the uncertainty of systems. Also, some fuzzy metrics have been proposed (for example, Ref.[2]), but there is no available measure for the purpose of the methodology. We proposed a metric between fuzzy points in Ref.[3] and a metric between fuzzy sets in Ref.[4–7]. But some faults were indicated for the metric between fuzzy sets. Therefore, we propose new metrics and show the properties in this paper. The metrics are available to analyze such systems.

First of all, some mathematical preparations for fuzzy sets and the axiom of metric are described. The next, a metric between fuzzy points is proposed and some examples and properties of the metric are shown. Moreover, a metric between fuzzy sets is proposed using the metric between fuzzy points and the some properties are shown.

In the second place, The modeling of uncertain systems for development of the mathematical method to analyze the risk of the systems is mentioned. The modeling is based on the proposed metric and supports the availability of the following discussion.

Last of all, the conclusion is described.

2. Metric

In this section, we propose a new metric between fuzzy points, and between fuzzy sets.

2.1. Preparation

In this paragraph, we define the fuzzy points and the fuzzy sets for main discussion.

Definition 2.1 (Fuzzy points) *The fuzzy point is a point with membership grade, which is real value in the closed interval* $[0,1]$. *In this paper, we denote the fuzzy point, which is the point x in X with the membership grade $a \in [0,1]$, as x^a.*

On this standpoint of Definition 2.1, a crisp point x is the fuzzy point x^1.

Definition 2.2 (Fuzzy sets[1]) *The fuzzy set A in X is a set characterized by the membership function $\mu_A : X \to [0,1]$.*

From the view of Definition 2.2, the fuzzy set A is union of fuzzy points, that is,

$$A = \bigcup_{a \le \mu_A(x)} x^a$$

Then, we will define a new metric between fuzzy points in the next paragraph. Before beginning of the discussion, we note that any metric satisfies the following axiom of metric:

Definition 2.3 (Axiom of metric) *Let X be a linear topological space. For any points x, y and z in X, if a real function $m : X \to \mathbf{R}$ satisfies the following conditions:*

(M1) $$\begin{cases} m(x,y) \ge 0 \\ m(x,y) = 0 \Leftrightarrow x = y \end{cases}$$

(M2) $$m(x,y) = m(y,x)$$

(M3) $$m(x,z) \le m(x,y) + m(y,z)$$

m is called metric on X, and the space X in which metric is defined, is called metric space.

The metric d between fuzzy points also must satisfy the above conditions. That is

$$\begin{cases} d(x^a, y^b) \geq 0 \\ d(x^a, y^b) = 0 \Leftrightarrow x^a = y^b \end{cases}$$

$$d(x^a, y^b) = d(y^b, x^a)$$

$$d(x^a, z^c) \leq d(x^a, y^b) + d(y^b, z^c)$$

Here,

$$x^a = y^b \Longleftrightarrow x = y \quad \text{and} \quad a = b$$

But, the sense of the membership grade of fuzzy points is not only the real value in the closed interval $[0, 1]$, but also likelihood. So that, the metric should satisfy not only the above conditions but also more concrete ones. The other hand, if the conditions is too strict, these are not applicable on a wide scale. From these view, the conditions are desirable to include the presented metrics until now.

We proposed a metric between fuzzy points and applied to analyze the recognition systems in Ref.[3]. Moreover, we proposed a metric between fuzzy sets and tried to applied to the fluctuation analysis of fuzzy control systems in Ref.[4–7]

In this paper, we propose a new metric between fuzzy points, and define one between fuzzy sets based on that.

2.2. Metric between fuzzy points

To define of the metric between fuzzy points, we introduce the following function N:

Definition 2.4 *For any real values p and q in $[0, \infty)$, the function $N : [0, \infty) \times [0, \infty) \to \mathbf{R}$ satisfies the following conditions.*

(N1) $\begin{cases} N(p, q) \geq 0 \\ N(p, q) = 0 \Rightarrow p = q \end{cases}$

(N2) $N(p, q) = N(q, p)$

(N3) $N(p, 0) = p$

(N4) $p \leq q \Rightarrow N(p, r) \leq N(q, r)$

(N5) $N((p + q), (r + s)) \leq N(p, r) + N(q, s)$

Example 2.1 *We show some examples of the function N.*

- $N_{\max}(p, q) = \max\{p, q\} \quad (p, q \geq 0)$
- $N_{+}(p, q) = p + q \quad (p, q \geq 0)$
- $N_{\oplus}(p, q) = p + q - pq \quad (0 \leq p, q \leq 1)$
- $N_2(p, q) = \sqrt{p^2 + q^2} \quad (p, q \geq 0)$

Now, we propose a metric between fuzzy points using the function N in Definition 2.4.

Definition 2.5 (Metric between fuzzy points) *Let m_1 and m_2 be the metrics on a linear topological space X and the closed interval $[0, 1]$, respectively. For any fuzzy points x^a and y^b, the metric d between the fuzzy points is defined as follows:*

$$d(x^a, y^b) = N(m_1(x, y), m_2(a, b)) \tag{1}$$

We show that the metric between fuzzy points satisfies the axiom of metric.

Theorem 2.1 *The metric between fuzzy points (1) defined in Definition 2.5 satisfies the axiom of metric (Definition 2.3).*

Proof *First of all, we show the metric satisfies* **(M1)**. *$d(x^a, y^b) \geq 0$ is obvious from* **(N1)**. *Moreover, we get*

$$\begin{aligned} d(x^a, y^b) = 0 \quad &\Leftrightarrow \quad N(D_1(x, y), D_2(a, b)) = 0 \\ &\Rightarrow \quad D_1(x, y) = D_2(a, b) = 0 \\ &\Leftrightarrow \quad x = y \text{ and } a = b \\ &\Leftrightarrow \quad x^a = y^b \end{aligned}$$

Conversely, if $x^a = y^b$ then $m_1(x, y) = m_2(a, b) = 0$. So $N(m_1(x, y), m_2(a, b)) = 0$ from **(N3)**, *that is $d(x^a, y^b) = 0$.*

In the second place, we show the metric satisfies **(M2)**. *From* **(M2)**, *we have*

$$\begin{aligned} d(x^a, y^b) &= N(m_1(x, y), m_2(a, b)) \\ &= N(m_1(y, x), m_2(b, a)) \\ &= d(y^b, x^a) \end{aligned}$$

Last of all, we show the metric satisfies **(M3)**. *We know*

$$d(x^a, z^c) = N(m_1(x, z), m_2(a, c))$$

From **(M3)**, *we get*

$$\begin{cases} m_1(x, z) \leq m_1(x, y) + m_1(y, z) \\ m_2(a, c) \leq m_2(a, b) + m_2(b, c) \end{cases}$$

Thus, we find

$$\begin{aligned} &N(m_1(x, z), m_2(a, c)) \\ &\quad \leq N(m_1(x, y) + m_1(y, z), \\ &\qquad\quad m_2(a, b) + m_2(b, c)) \\ &\quad \leq N(m_1(x, y), m_2(a, b)) \\ &\qquad\quad + N(m_1(y, z), m_2(b, c)) \\ &\quad = d(x^a, y^b) + d(y^b, z^c) \end{aligned}$$

Theorem 2.2 *For any crisp points x and y in X,*

$$d(x^1, y^1) = m(x, y)$$

Proof *We find*

$$\begin{aligned} d(x^1, y^1) &= N(m(x, y), m'(1, 1)) \\ &= N(m(x, y), 0) \\ &= m(x, y) \end{aligned}$$

Example 2.2 *We show some examples of the metric between fuzzy points.*

- $$d_{\max}(x^a, y^b) = \max\{\sum_{i=1}^{n} |x_i - y_i|,\ |a-b|\}$$
  Here, $x = {}^t(x_1, \ldots, x_n)$*,* $y = {}^t(y_1, \ldots, y_n)$*.*
- $$d_2(x^a, y^b) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2 + (a-b)^2}$$
  Here, $x = {}^t(x_1, \ldots, x_n)$*,* $y = {}^t(y_1, \ldots, y_n)$*.*
- $$d_p(x^a, y^b) = \left(\int_0^1 |x(t) - y(t)|^p \mathrm{d}t\right)^{1/p} + w|a-b|$$
  Here, $x = x(t)$ *and* $y = y(t)$ *are both the functions from* $\mathbf{R}$ *to itself, and* w *is a positive constant.*

2.3. Metric between fuzzy sets

In this paragraph, we will define the metric between fuzzy sets based on the metric between fuzzy points. First of all, we define the metric between a fuzzy point and a fuzzy set.

Definition 2.6 *Let* x^a *and* A *be a fuzzy point and a fuzzy set on the linear topological space* X*, respectively. The metric* d *between fuzzy point* x^a *and* A *is defined as follows:*

$$d(x^a, A) = \inf_{\xi^\alpha \in A} d(x^a, \xi^\alpha) \tag{2}$$

Here,

$$\xi^\alpha \in A \Longleftrightarrow \alpha \le \mu_A(\xi)$$

Then, we introduce the metric between fuzzy sets using the metric between the fuzzy point and the fuzzy set (2) in Definition 2.6.

Definition 2.7 (Metric between fuzzy sets) *Let* A *and* B *be both fuzzy sets on the linear topological space* X*. The metric* D *between fuzzy sets* A *and* B *is defined in the sense of Hausdorff metric as follows:*

$$D(A, B) = \sup_{x^a \in A} d(x^a, B) \vee \sup_{y^b \in B} d(y^b, A) \tag{3}$$

We find that the metric between fuzzy sets satisfies the axiom of metric.

Theorem 2.3 *The metric between fuzzy sets (3) defined in Definition 2.7 satisfies the axiom of metric (Definition 2.3).*

Proof *First of all, we show the metric satisfies* **(M1)***.* $D(A, B) \ge 0$ *is obvious from* **(N1)***. Moreover, we get*

$$\begin{aligned} & D(A, B) = 0 \\ \Leftrightarrow\ & \sup_{x^a \in A} d(x^a, B) = \sup_{y^b \in B} d(y^b, A) = 0 \end{aligned}$$

Here, we find

$$\begin{aligned} & \sup_{x^a \in A} d(x^a, B) = 0 \\ \Leftrightarrow\ & \forall x^a \in A,\ d(x^a, B) = 0 \\ \Leftrightarrow\ & \forall x^a \in A,\ \inf_{\xi^\alpha \in B} d(x^a, \xi^\alpha) = 0 \\ \Leftrightarrow\ & \forall x^a \in A,\ \exists \xi^\alpha \in B,\ d(x^a, \xi^\alpha) = 0 \\ \Leftrightarrow\ & \forall x^a \in A,\ \exists \xi^\alpha \in B,\ x^a = \xi^\alpha \\ \Leftrightarrow\ & \forall x^a \in A,\ a \le \mu_B(x) \\ \Leftrightarrow\ & \forall x,\ \mu_A(x) \le \mu_B(x) \\ \Leftrightarrow\ & A \subseteq B \end{aligned}$$

In the same way, we have

$$\sup_{y^b \in B} d(y^b, A) = 0 \Leftrightarrow B \subseteq A$$

Therefore, we get

$$\begin{aligned} & D(A, B) = 0 \\ \Leftrightarrow\ & A \subseteq B \text{ and } B \subseteq A \\ \Leftrightarrow\ & \forall x,\ \mu_A(x) \le \mu_B(x) \text{ and } \mu_B(x) \le \mu_A(x) \\ \Leftrightarrow\ & A = B \end{aligned}$$

In the second place, it is obvious that the metric satisfies **(M2)***, that is,* $D(A, B) = D(B, A)$*.*

Last of all, we show the metric satisfies **(M3)***. For any fuzzy points* $x^a \in A$*,* $y^b \in B$ *and* $z^c \in C$*, We know*

$$d(x^a, z^c) \le d(x^a, y^b) + d(y^b, z^c)$$

Since z^c *is an arbitrary point in* C*, we get*

$$d(x^a, C) \le d(x^a, y^b) + d(y^b, C)$$

Moreover, since y^b *is an arbitrary point in* B*, we have*

$$d(x^a, C) \le d(x^a, B) + \sup_{y^b \in B} d(y^b, C)$$

Furthermore, since x^a *is also an arbitrary point in* A*, we get*

$$\sup_{x^a \in A} d(x^a, C) \le \sup_{x^a \in A} d(x^a, B) + \sup_{y^b \in B} d(y^b, C) \tag{4}$$

From Eq. (4), we find

$$\begin{aligned} & D(A, C) \\ =\ & \sup_{x^a \in A} d(x^a, C) \vee \sup_{z^c \in C} d(z^c, A) \\ \le\ & (\sup_{x^a \in A} d(x^a, B) + \sup_{y^b \in B} d(y^b, C)) \\ & \quad \vee (\sup_{z^c \in C} d(z^c, B) + \sup_{y^b \in B} d(y^b, A)) \\ \le\ & (\sup_{x^a \in A} d(x^a, B) \vee \sup_{y^b \in B} d(y^b, A)) \\ & \quad + (\sup_{y^b \in B} d(y^b, C) \vee \sup_{z^c \in C} d(z^c, B)) \\ =\ & D(A, B) + D(B, C) \end{aligned}$$

3. Modeling of Uncertain Systems

In this section, we mention the modeling of uncertain systems for risk analysis. First of all, we define the fuzzy mapping and the fuzzy set-valued mapping. In the second place, we describe the modeling of the uncertain systems using the fuzzy mappings.

3.1. Fuzzy mapping

Let X and Y be both the linear topological spaces, $\mathcal{P}(X)$ be a family of all fuzzy points with the positive membership grade on X and $\mathcal{F}(X)$ be a family of all fuzzy sets on X including fuzzy points. We consider the function $g : X \to Y$. The fuzzy mapping $f : \mathcal{P}(X) \to \mathcal{P}(Y)$ is defined as follows:

$$\operatorname{supp} f(x^a) = g(x)$$

$$\mu_{f(x^a)}(y) = \begin{cases} \sup\limits_{x \in g^{-1}(y)} \mu_{x^a}(x) & (g^{-1}(y) \neq \phi) \\ 0 & (g^{-1}(y) = \phi) \end{cases}$$

Here, $\operatorname{supp} A$ means the support of the fuzzy set A. Moreover, The fuzzy set-valued mapping G is defined as follows:

$$G : \mathcal{P}(X) \to \mathcal{F}(X)$$

The fuzzy set-valued mapping is extension from the mapping in Ref.[4–7].

3.2. Modeling

As extension of the modeling the uncertain systems in Ref.[4–7], the correspondence between input signal and output response of such systems can be denoted by the following equations:

$$\xi^\alpha \in G(x^a, f(x^a)) \tag{5}$$

Here, $f : \mathcal{P}(X) \to \mathcal{P}(Y)$ is the fuzzy mapping which represents the original (certain) system, and $G : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{F}(X)$ is the fuzzy set-valued mapping which represents the uncertainty of system. This equation means

$$\alpha \leq \mu_{G(x^a, f(x^a))}(\xi)$$

In the case to consider of the β-level likelihood, we introduce the metric d_β between fuzzy points instead of d as follws:

$$d_\beta(x^a, y^b) = \begin{cases} d(x^a, y^b) & (a, b \geq \beta) \\ +\infty & (\text{otherwise}) \end{cases}$$

and the metric D_β between fuzzy sets, which can be defined using d_β in place of d. In this case, Eq. (5) can be re-formulated as follows:

$$\xi^\alpha \in_\beta G(x^a, f(x^a)) \tag{6}$$

When $\beta = 0$, d_β and Eq. (6) result in d and Eq. (5) respectively, that is, d_β and Eq. (6) are generalization of d and Eq. (5) respectively. If the equation has the solution x^{a*} such that

$$x^{a*} \in_\beta G(x^{a*}, f(x^{a*}))$$

x^{a*} is regarded as one of the tolerable responses of the uncertain system. Furthermore, if we find the existence conditions of x^{a*}, we can consider that the risk of uncertainty of the system has estimated. The fuzzy metrics are necessary to discuss the above discussion.

4. Conclusion

In this paper, we proposed new metrics to analyzing the risk of such uncertain systems and showed some properties and examples of the metrics. Moreover, we introduced the fuzzy mapping and the fuzzy set-valued mapping, and described the modeling of the uncertain systems using these mappings.

In the forthcoming paper, we will introduce the existence conditions of the tolerable responses of the uncertain systems.

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