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Necessary and Sufficient Condition for Two-Cell CNNs with Space-Invariant Connections to be Globally Stable

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1. Introduction

Global stability analysis is one of the most fundamental research topics in the study of Cellular Neural Networks (CNNs) [1] and other nonlinear dynamical systems. A CNN is said to be globally stable if its state converges to some equilibrium point for any initial condition. Although a large number of papers concerning the global stability of CNNs have been published so far, the relation between the global stability of CNNs and network parameters is not completely clarified. Therefore, as the first step toward the complete understanding of complex behavior of CNNs, it is important to investigate thoroughly the dynamical behavior of two-cell CNNs.

The first study of the global dynamical behavior of two-cell CNNs was made by Civalleri and Gilli [2]. They have considered two-cell CNNs described by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{1}$$

where the value of y_i, which represents the output of the i-th cell, is determined by the value of x_i, which represents the state of the i-th cell, via the piecewise linear function $y_i = f_i(x_i) \triangleq \frac{1}{2}(|x_i+1| - |x_i-1|)$ for $i = 1, 2$. Under the assumptions that $a_{11} > 1$ and $a_{22} > 1$, they have studied in detail the location of equilibrium points, the domain of attraction of each equilibrium point, and the global stability of the two-cell CNNs. Although most of their results are correct and valuable, it has recently been pointed out that the global stability analysis for the case where $a_{12}a_{21} < 0$ is not correct [3]. This means that even for such simple CNNs the global stability is not completely understood.

In this paper, we investigate the global dynamical behavior of two-cell CNNs described by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} s & p \\ -q & s \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{2}$$

where the parameters s, p and q are assumed to satisfy

$$s > 1, \quad p > 0 \quad \text{and} \quad q > 0. \tag{3}$$

These CNNs correspond to the special case of (1) where $a_{11} = a_{22} > 1$ and $a_{12} > 0 > a_{21}$. The former condition is general in the sense that the coupling coefficients are often assumed to be space-invariant. Also, in the case where $a_{12}a_{21} < 0$ we can assume $a_{12} > 0$ without loss of generality. In this paper, the necessary and sufficient condition for a CNN (2) with (3) to be globally stable will be given. Since it has been already known that a two-cell CNN (1) is globally stable if $a_{12}a_{21} \geq 0$, the results of this paper completes the global stability analysis of (1) for the case where $a_{11} = a_{22}$.

2. Main Results

The following theorem is a main result of this paper.

Theorem 1 *A two-cell CNN (2) with (3) is globally stable if the parameters s, p, and q satisfy the following three conditions:*

$$\min\{p, q\} \leq s - 1 \tag{4}$$

$$\frac{s-1}{\sqrt{pq}} \exp\left\{\frac{\pi}{2} \cdot \frac{s-1}{\sqrt{pq}}\right\} < 1 \tag{5}$$

$$\exp\left\{\frac{2(s-1)}{\sqrt{pq}} \tan^{-1}\left(\frac{s}{\sqrt{pq}}\right)\right\} > \frac{pq}{(s-1)^2(s^2+pq)} \tag{6}$$

Combining Theorem 1 with the authors' previous results [3], we obtain the following theorem:

Theorem 2 *A two-cell CNN (2) with (3) is globally stable if and only if the parameters s, p and q satisfy (4) and (6).*

Equation (6) prescribes the relation between s and $\sqrt{pq}$. The parameter regions for global stability and instability in the case where $s - 1 \geq \min\{p, q\}$ are illustrated in Fig.1.

3. Proof of Theorem 1

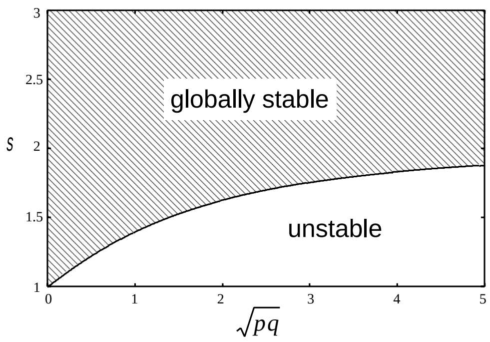

Figure 1: Parameter region for global stability in the case where $s-1 \geq \min\{p, q\}$

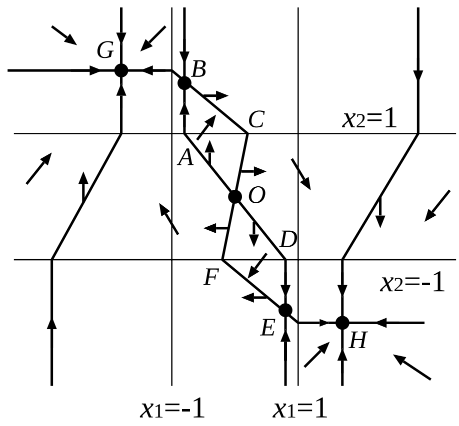

Figure 2: Phase portrait of a two-cell CNN satisfying (4) – (6) and $p < q$.

3.1. Phase Portrait

The phase portrait of a two-cell CNN satisfying (4) – (6) and $p < q$ (we can assume this without loss of generality) is shown in Fig.2 where nullclines, the direction of the flow and five equilibrium points $O : (0, 0)$, $B : (-\frac{p}{s-1}, s+\frac{pq}{s-1})$, $E : (\frac{p}{s-1}, -s-\frac{pq}{s-1})$, $G : (-s+p, s+q)$ and $H : (s-p, -s-q)$ are indicated. By investigating the eigenvalues of the Jacobian matrix at each equilibrium point, it is easily seen that O is an unstable focus, B and E are saddle points, and G and H are stable nodes. Moreover, it is known that the phase portrait has the following important characteristics (see [3] for detail).

(i) The trajectory passing through the point C crosses the line segment OD (see Fig.3).

(ii) There exists a trajectory connecting two equilibrium points B and H (see Fig.3). Moreover, this trajectory passes through the point $(p+\frac{s(s-1)}{q}, 1)$.

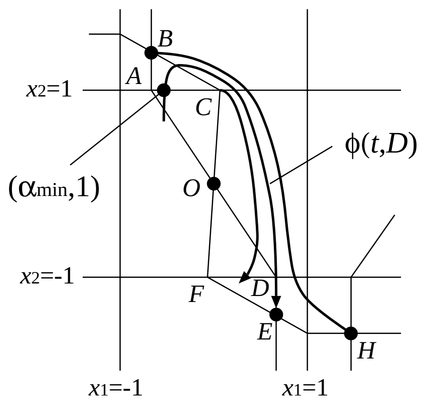

Figure 3: The trajectory $\phi(t, D)$ and the point $(\alpha_{\min}, 1)$

From Fig.2 we can easily see that there may exist a limit cycle crossing the line segment OA, AC, BC, OD, DF and EF consecutively. We also see that if such a limit cycle does not exist then any trajectory converges to one of the five equilibrium points, that is, the CNN is globally stable. Let $\phi(t, D)$ be the trajectory passing through the point D at $t = 0$. Then, due to the characteristics (i) and (ii) given above, $\phi(t, D)$ crosses the line segment AC and BC before reaching D as shown in Fig.3. Therefore, if there exists a limit cycle, it must cross the line segment AC at some point between $(\alpha_{\min}, 1)$, the intersection of $\phi(t, D)$ and the line segment AC, and $C : (\frac{s-1}{q}, 1)$ since otherwise it will converge to either E or H, which is a contradiction. We thus hereafter concentrate our attention upon the trajectory $\phi(t, (\alpha, 1))$ where

$$-\frac{p}{s-1} < \alpha_{\min} < \alpha < \frac{s-1}{q}. \tag{7}$$

Let $(\beta, 1)$ be the point where $\phi(t, (\alpha, 1))$ with (7) intersects with the line $x_2 = 1$. Then, it is obvious that

$$\frac{s-1}{q} < \beta < \beta_{\max} < p + \frac{s(s-1)}{q} \tag{8}$$

where $\beta_{\max}$ is the value of β corresponding to $\alpha = \alpha_{\min}$. Moreover, let $U = (u_1, u_2)$ and $V = (v_1, v_2)$ be the points where $\phi(t, (\alpha, 1))$ satisfying (7) intersects with the line segment OA and OD, respectively (see Fig.4).

3.2. Analysis of the Trajectory $\phi(t, (\alpha, 1))$

First we investigate the relation between α and β.

Lemma 1 *For each α satisfying (7), β is uniquely de-*

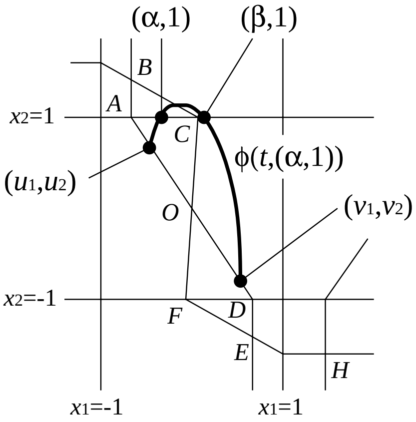

Figure 4: The trajectory $\phi(t,(\alpha,1))$

termined as a solution of the equation $g(\alpha,\beta)$ *where*

$$
\begin{aligned}
g(w,z) \triangleq (w+\frac{p}{s-1})\left\{-w+p+\frac{s(s-1)}{q}\right\}^{s-1} \\
-(z+\frac{p}{s-1})\left\{-z+p+\frac{s(s-1)}{q}\right\}^{s-1} \qquad (9)
\end{aligned}
$$

Lemma 2 *If* $s \geq 2$ *then* $\beta^2 > \alpha^2$ *for any* α *satisfying (7).*

Proof Let $g_1(w,z) \triangleq z^2 - w^2$. Since $\beta \to p + s(s-1)/q$ as $\alpha \to -p/(s-1)$, we have

$$
\begin{aligned}
&\lim_{\alpha \to -p/(s-1)} g_1(\alpha,\beta) \\
&= \left\{p+\frac{s(s-1)}{q}\right\}^2 - \left\{-\frac{p}{s-1}\right\}^2 > 0. \qquad (10)
\end{aligned}
$$

Also, since $\beta \to (s-1)/q$ as $\alpha \to (s-1)/q$, we have

$$\lim_{\alpha \to (s-1)/q} g_1(\alpha,\beta) = 0 \qquad (11)$$

Moreover, by the implicit function theorem, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}g_1(\alpha,\beta)}{\mathrm{d}\alpha} \\
&= 2\beta \cdot \frac{\mathrm{d}\beta}{\mathrm{d}\alpha} - 2\alpha \\
&= -2\beta \cdot \frac{(\frac{s-1}{q}-\alpha)\left\{-\alpha+p+\frac{s(s-1)}{q}\right\}^{s-2}}{(\beta-\frac{s-1}{q})\left\{-\beta+p+\frac{s(s-1)}{q}\right\}^{s-2}} - 2\alpha \\
&< -2\beta \cdot \frac{\frac{s-1}{q}-\alpha}{\beta-\frac{s-1}{q}} - 2\alpha \\
&= -\frac{2(s-1)}{q} \cdot \frac{\beta-\alpha}{\beta-\frac{s-1}{q}} < 0. \qquad (12)
\end{aligned}
$$

It follows from (10)–(12) that the function $g_1(\alpha,\beta)$ is positive for all α satisfying (7) if $s \geq 2$. ■

Lemma 3 *If* $s < 2$ *then*

$$\beta \geq (s-1)\left(\frac{s}{q}-\alpha\right) \qquad (13)$$

for any α *satisfying (7).*

Proof Let $g_2(w) \triangleq (s-1)(s/q-w)$. Since β and $g_2(\alpha)$ take the same value for $\alpha = (s-1)/q$ and

$$\left.\frac{\mathrm{d}\beta}{\mathrm{d}\alpha}\right|_{\alpha=\frac{s-1}{q}} = -1 < -(s-1) = \left.\frac{\mathrm{d}g_2(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=\frac{s-1}{q}}$$

holds, there exists a positive number ϵ_0 such that $\beta > g_2(\alpha)$ for $(s-1)/q - \epsilon_0 < \alpha < (s-1)/q$. Assume now that $\beta < g_2(\alpha)$ for some α satisfying $-p/(s-1) < \alpha \leq (s-1)/q - \epsilon_0$. Then there must exist an α such that

$$\beta = g_2(\alpha) \quad \text{and} \quad \frac{\mathrm{d}\beta}{\mathrm{d}\alpha} > -(s-1). \qquad (14)$$

However, since

$$
\begin{aligned}
&\left.\frac{\mathrm{d}\beta}{\mathrm{d}\alpha}\right|_{\beta=g_2(\alpha)} \\
&= -(s-1)\left\{1+\frac{s(2-s)}{q(s-1)^2}\cdot\frac{(s-1)^2+pq}{-\alpha+p+\frac{s(s-1)}{q}}\right\} \\
&< -(s-1)
\end{aligned}
$$

holds if $s < 2$ and $-p/(s-1) < \alpha < (s-1)/q$, there is no α which satisfies (14). This is a contradiction. ■

Next we investigate the relation between u_2 (resp., v_2) and α (resp., β).

Lemma 4 *The value of* u_2 *is expressed in terms of* α *as* $u_2(\alpha) = \gamma(\alpha)\,\sigma_1(\alpha)$ *where*

$$
\begin{aligned}
\gamma(w) &\triangleq \frac{(s-1)\sqrt{p+qw^2}}{\sqrt{p\{(s-1)^2+pq\}}} \\
\sigma_1(w) &\triangleq \exp\left\{-\frac{s-1}{\sqrt{pq}}\arctan\left(\frac{s-1}{\sqrt{pq}}\cdot\frac{w+\frac{p}{s-1}}{\frac{s-1}{q}-w}\right)\right\}
\end{aligned}
$$

Lemma 5 *The value of* v_2 *is expressed in terms of* β *as* $v_2(\beta) = -\gamma(\beta)\,\sigma_2(\beta)$ *where*

$$\sigma_2(z) \triangleq \exp\left\{\frac{s-1}{\sqrt{pq}}\arctan\left(\frac{s-1}{\sqrt{pq}}\cdot\frac{z+\frac{p}{s-1}}{z-\frac{s-1}{q}}\right)\right\}$$

Let us now define the function $h(w,z)$ as

$$h(w,z) \triangleq \gamma(z)\,\sigma_2(z) - \gamma(w)\,\sigma_1(w).$$

Taking the symmetry of the flow into account, we have the following lemma:

Lemma 6 *A limit cycle exists if and only if* $h(\alpha, \beta)$ *has a zero in the interval determined by (7).*

The following lemma follows from Lemmas 2 – 5.

Lemma 7 $h(\alpha, \beta)$ *is positive for any* α *satisfying (7).*

Proof In the case where $s \geq 2$, we can easily see from Lemma 2 that $h(\alpha, \beta) > 0$ for any α. Thus it suffices for us to show that $h(\alpha, \beta) > 0$ for $s < 2$. Let $h_1(w, z) \triangleq \{\gamma(z)\sigma_2(z)\}^2 - \{\gamma(w)\sigma_1(w)\}^2$. It is obvious that $h(\alpha, \beta) > 0$ if and only if $h_1(\alpha, \beta) > 0$. Also, since β satisfies (13) and $\{\gamma(\beta)\sigma_2(\beta)\}^2$ is monotone increasing with respect to β in the interval defined by (8),

$$h_1(\alpha, \beta) \geq h_1\left(\alpha, (s-1)\left(\frac{s}{q} - \alpha\right)\right) \triangleq h_2(\alpha)$$

holds for all α satisfying (7). We will show in the following that $h_2(\alpha)$ is positive for

$$-\frac{p}{s-1} < \alpha < \frac{s-1}{q} \tag{15}$$

which implies the assertion of this lemma holds true. Note here that the function $h_2(\alpha)$ is defined for all α in the interval (15) since $\gamma(\cdot)$, $\sigma_1(\cdot)$ and $\sigma_2(\cdot)$ are defined for all real numbers. Since $(s-1)(s/q - \alpha) \to (s-1)/q$ as $\alpha \to (s-1)/q$, we have

$$\lim_{\alpha \to (s-1)/q} h_2(\alpha) = \lim_{\substack{w \to (s-1)/q - 0 \\ z \to (s-1)/q + 0}} h_1(w, z) > 0. \tag{16}$$

Also, since $(s-1)(s/q - \alpha) \to p + s(s-1)/q$ as $\alpha \to -p/(s-1)$, we have from (6)

$$\begin{aligned}
&\lim_{\alpha \to -p/(s-1)} h_2(\alpha) \\
&= h_1\left(-\frac{p}{s-1}, p + \frac{s(s-1)}{q}\right) \\
&= \gamma\left(p + \frac{s(s-1)}{q}\right)^2 \sigma_2\left(p + \frac{s(s-1)}{q}\right)^2 \\
&\quad - \gamma\left(-\frac{p}{s-1}\right)^2 \sigma_1\left(-\frac{p}{s-1}\right)^2 \\
&= \gamma\left(p + \frac{s(s-1)}{q}\right)^2 \sigma_2\left(p + \frac{s(s-1)}{q}\right)^2 - 1 \\
&= \frac{(s-1)^2(s^2 + pq)}{pq} \exp\left\{\frac{2(s-1)}{\sqrt{pq}} \tan^{-1}\left(\frac{s}{\sqrt{pq}}\right)\right\} - 1 \\
&> 0.
\end{aligned} \tag{17}$$

By differentiating $h_2(\alpha)$ with respect to α, we obtain

$$\begin{aligned}
\frac{\mathrm{d}h_2(\alpha)}{\mathrm{d}\alpha} &= -\frac{2(s-1)^2\{(s-1) - q\alpha\}}{p\{(s-1)^2 + pq\}} \\
&\quad \times \left\{(s-1)^2\sigma_2\left((s-1)\left(\frac{s}{q} - \alpha\right)\right) - \sigma_1(\alpha)\right\}
\end{aligned}$$

The right-hand side of this equation vanishes if and only if α satisfies

$$\sigma_2\left((s-1)\left(\frac{s}{q} - \alpha\right)\right) = \frac{\sigma_1(\alpha)}{(s-1)^2}. \tag{18}$$

Substituting (18) into $h_2(\alpha)$, we have

$$h_2(\alpha)\Big|_{\frac{\mathrm{d}h_2(\alpha)}{\mathrm{d}\alpha} = 0} = s\left\{\frac{s}{q} + \frac{2-s}{(s-1)^2}p - 2\alpha\right\}\sigma_1(\alpha).$$

If $\mathrm{d}h_2(\alpha)/\mathrm{d}\alpha$ does not vanish in the interval (15) then it is obvious from (16) and (17) that $h_2(\alpha)$ is positive for all α in (15). If $\mathrm{d}h_2(\alpha)/\mathrm{d}\alpha$ vanishes for $\alpha = \alpha^*$ in the interval (15) then $h_2(\alpha)$ is also positive for all α since

$$\begin{aligned}
h_2(\alpha^*) &> s\left\{\frac{s}{q} + \frac{2-s}{(s-1)^2}p - \frac{2(s-1)}{q}\right\}\sigma_1(\alpha^*) \\
&= s(2-s)\left\{\frac{1}{q} + \frac{p}{(s-1)^2}\right\}\sigma_1(\alpha^*) \\
&> 0
\end{aligned}$$

holds for any $\alpha^* < (s-1)/q$, where the last inequality is derived from the assumption that $s < 2$. ∎

Lemmas 6 and 7 completes the proof of Theorem 1.

4. Conclusions

We have obtained the necessary and sufficient condition for two-cell CNNs with space-invariant and opposite-sign connections to be globally stable. Extension of the result given in this paper to the case where two-cell CNNs are space-varying and/or have non-zero bias is the future problem.

Acknowledgments

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