

Workers versus Capitalists: A Differential Game Approach to Goodwin's Growth Cycle Model

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1. Introduction

Goodwin's growth cycle model [2] describes the symbiosis – partly complementary, partly hostile – of workers and capitalists. Goodwin's system generates the closed orbits, and gives us an endogenous explanation of both growth and cycle. Our aim is to give the theoretical foundation for the optimality of such cycles. In other words, the purpose of this note is to show that optimal cyclical behaviour may be existed in the framework similar to Goodwin's one. To do this, we approach to Goodwin's model by attempting to introduce the interacting agents, namely workers and capitalists, who behave rationally into the model. Our procedure is as follows. First, we formulate this problem as a differential game between workers and capitalists. Second, we derive a open-loop Nash solution of the differential game. Finally, we demonstrate the existence of a limit cycle by using the Hopf bifurcation theorem and the numerical method.

Although a large number of studies have been made on the optimality of limit cycle in optimal control models, there are few papers on that in differential game models. Our differential game model is constructed on the basis of the seminal paper by Feichtinger and Novak [1].

2. Goodwin's Model

In this section, we introduce Goodwin's model briefly. Goodwin put the following assumptions:

Assumption 2.1. steady technical progress (disembodied).

Assumption 2.2. steady growth in the labour force.

Assumption 2.3. only two factors of production, labour and 'capital' (plant and equipment), both homogeneous and non-specific.

Assumption 2.4. all quantities real and net.

Assumption 2.5. all wages consumed, all profits saved and invested.

Assumption 2.6. a constant capital-output ratio.

Assumption 2.7. a real wage rate which rises in the neighbourhood of full employment.

Goodwin explained that assumptions 2.1–2.5 are made for convenience and assumptions 2.6 and 2.7 are of a more empirical and disputable sort.

Symbols :

q = output.

k = capital.

w = wage rate.

n = labour supply.

l = employment.

$a = q/l$ = labour productivity.

$\sigma = k/q$ = capital-output ratio.

$v = l/n$ = employment rate.

$u = wl/q$ = workers' share of income.

Assumption 2.1 denotes $a = a(0) \exp(\alpha t)$, namely

$$\frac{\dot{a}}{a} = \alpha, \quad \alpha \geq 0, \quad (1)$$

and assumption 2.2 represents $n = n(0) \exp(\beta t)$, that is

$$\frac{\dot{n}}{n} = \beta, \quad \beta \geq 0. \quad (2)$$

Moreover, from assumption 2.5,

$$\dot{k} = (1 - u)q, \quad (3)$$

and from assumption 2.6,

$$\frac{k}{q} = \sigma, \quad \sigma > 0. \quad (4)$$

Furthermore, assumption 2.7 is written as

$$\frac{\dot{w}}{w} = f(v) \cong -\gamma + \rho v. \quad (5)$$

Thus, From equations (1)– (5), we get the following dynamical system:

$$\dot{v} = \left[\frac{1-u}{\sigma} - (\alpha + \beta) \right] v, \quad (6)$$

$$\dot{u} = [-(\alpha + \gamma) + \rho v]u. \quad (7)$$

Equations (6), (7) are formally the same as that of Volterra-Lotka equation in mathematical biology. As a result, it follows that system (6), (7) generates the closed orbits.

3. Differential Game Model

3.1. Model

The first purpose of this section is to formulate the problem similar to Goodwin's one as a differential game between workers and capitalists. Note that we do not attach importance to maintain Goodwin's original algebraic structure. Therefore, although the qualitative property of Goodwin's system is not maintained, the essence of Goodwin's model "symbiosis and conflict between workers and capitalists (perpetual cycles of v and u)" is preserved, and the interpretation that such fluctuations may result from non-optimal behaviour of the agents is ruled out.

To begin with, we modify Goodwin's original assumptions 2.5 and 2.7 as follows (and maintain all assumptions except these):

Assumption 3.1. the percentage c of wages consumed, all profits saved and invested.

Assumption 3.2. workers' share of income is decided by capitalists.

Consequently, the system is given by

$$\dot{v} = \left(\frac{1-cu}{\sigma} - \kappa \right) v. \quad (8)$$

Here, $\kappa = \alpha + \beta$ denotes natural growth rates.

As Goodwin had already pointed out that the modification to assumption 3.1 has no effect on the qualitative property of Goodwin's system. In contrast, the alteration to assumption 3.2 has serious effect on that. This is because we give priority to analytical convenience. However, there is Lancaster's model [3] which uses a system of one dimensional differential equation in a differential game between workers and capitalists as we do, although the state variable in Lancaster's model is capital stock.

Next, we formulate the objective functional of each players.

Workers (player 1) draw utility from the workers' share of consumption C ($= cu$) and the employment rate. On the other hand, changes of the propensity to consume c cause costs.

Thus, the objective functional of workers is expressed as follows:

$$J_1 = \int_0^\infty \exp(-\rho_1 t) [F_1(C, v) - C_1(c)] dt. \quad (9)$$

Here, ρ_1 is the positive discount rate of workers. We assume that

$$\begin{aligned} \frac{\partial F_1}{\partial C} &> 0, & \frac{\partial F_1}{\partial v} &> 0, & \frac{\partial^2 F_1}{\partial C^2} &\leq 0, & \frac{\partial^2 F_1}{\partial v^2} &\leq 0, \\ \frac{dC_1}{dc} &> 0, & \frac{d^2 C_1}{dc^2} &> 0. \end{aligned}$$

Capitalists (player 2) derive utility from the capitalists' share of profit S ($= 1 - cu$) and the employment rate. As assumed above, capitalists decide workers' share of income u . This assumption does not imply, however, that capitalists unilaterally control workers' share of income. On the contrary, adjustments of workers' share of income are affected by workers' bargaining power and induce costs. Hence, the objective functional of capitalists is defined as follows:

$$J_2 = \int_0^\infty \exp(-\rho_2 t) [F_2(S, v) - C_2(u)] dt. \quad (10)$$

Here, ρ_2 is the positive discount rate of capitalists. Again, we assume that

$$\begin{aligned} \frac{\partial F_2}{\partial S} &> 0, & \frac{\partial F_2}{\partial v} &> 0, & \frac{\partial^2 F_2}{\partial S^2} &\leq 0, & \frac{\partial^2 F_2}{\partial v^2} &\leq 0, \\ \frac{dC_2}{du} &< 0, & \frac{d^2 C_2}{du^2} &> 0. \end{aligned}$$

Note that capitalists' cost function is not upward-sloping curve. It is costly for capitalists to decrease workers' share of income because of resistance of workers.

To simplify the analytical procedure, we specify the utility functions F_i and the cost functions C_i as

$$\begin{aligned} F_1(C, v) &= Cv = cuv, \\ F_2(S, v) &= Sv = (1 - cu)v, \\ C_1(c) &= c^{\theta_1}, \quad \theta_1 > 1, \\ C_2(u) &= u^{-\theta_2} - 1, \quad \theta_2 > 0. \end{aligned}$$

From these specifications, we may put a following interpretation on parameters θ_1, θ_2 . That is, θ_1 denotes the type of penalty for changes of the consumption rate. On the other hand, θ_2 represents the intensity of workers' bargaining power.

3.2. Mathematical Analysis

To find an open-loop Nash solution to our differential game, we make use of Pontryagin's maximum principle and

formulate the current value Hamiltonians for both players

$$\mathcal{H}_1 = cuv - c^{\theta_1} + \lambda \left(\frac{1-cu}{\sigma} - \kappa \right) v, \quad (11)$$

$$\mathcal{H}_2 = (1-cu)v - (u^{-\theta_2} - 1) + \mu \left(\frac{1-cu}{\sigma} - \kappa \right) v, \quad (12)$$

where λ and μ denotes the current value costates of v for workers and capitalists, respectively. And the maximum conditions are

$$\frac{\partial \mathcal{H}_1}{\partial c} = uv - \theta_1 c^{\theta_1-1} - \frac{\lambda uv}{\sigma} = 0, \quad (13)$$

$$\frac{\partial \mathcal{H}_2}{\partial u} = -cv + \theta_2 u^{-(\theta_2+1)} - \frac{\mu cv}{\sigma} = 0, \quad (14)$$

and

$$\dot{\lambda} = \lambda \rho_1 - \frac{\partial \mathcal{H}_1}{\partial v} = \lambda \left(\rho_1 - \frac{1-cu}{\sigma} + \kappa \right) - cu, \quad (15)$$

$$\dot{\mu} = \mu \rho_2 - \frac{\partial \mathcal{H}_2}{\partial v} = \mu \left(\rho_2 - \frac{1-cu}{\sigma} + \kappa \right) - (1-cu). \quad (16)$$

We get the following canonical system:

$$\dot{v} = \left(\frac{1-c^*u^*}{\sigma} - \kappa \right) v, \quad (17)$$

$$\dot{\lambda} = \lambda \left(\rho_1 - \frac{1-c^*u^*}{\sigma} + \kappa \right) - c^*u^*, \quad (18)$$

$$\dot{\mu} = \mu \left(\rho_2 - \frac{1-c^*u^*}{\sigma} + \kappa \right) - (1-c^*u^*), \quad (19)$$

where

$$c^*u^* = \left[v^{\theta_2-\theta_1} \theta_1^{-\theta_2} \theta_2^{\theta_1} \left(1 - \frac{\lambda}{\sigma} \right)^{\theta_2} \left(1 + \frac{\mu}{\sigma} \right)^{-\theta_1} \right]^{\frac{1}{\theta_1-\theta_2+\theta_1\theta_2}}. \quad (20)$$

The steady state of canonical system (17)–(19) are

$$\lambda^* = \frac{1-\kappa\sigma}{\rho_1}, \quad (21)$$

$$\mu^* = \frac{\kappa\sigma}{\rho_2}, \quad (22)$$

and the value of v^* can be found by the right-hand side of (20) = $1 - \kappa\sigma$ after substituting (21), (22) into (20). Here, note that equation (20) make sense only for $\lambda < \sigma$. Thus, the condition $\kappa > 1/\sigma - \rho_1$ has to be fulfilled.

3.3. Stability Analysis

To carry out a stability analysis, we linearize the canonical system. This yields

$$\begin{bmatrix} \dot{v} \\ \dot{\lambda} \\ \dot{\mu} \end{bmatrix} = J^* \begin{bmatrix} v - v^* \\ \lambda - \lambda^* \\ \mu - \mu^* \end{bmatrix}, \quad (23)$$

where J^* is the Jacobian matrix evaluated the steady state

$$\begin{bmatrix} \frac{(\theta_1-\theta_2)(1-\kappa\sigma)}{(\theta_1-\theta_2+\theta_1\theta_2)\sigma} & \frac{v^*\theta_2\rho_1(1-\kappa\sigma)}{(\theta_1-\theta_2+\theta_1\theta_2)\sigma(\rho_1\sigma-(1-\kappa\sigma))} \\ \frac{(\theta_1-\theta_2)(1-\kappa\sigma)(\rho_1\sigma-(1-\kappa\sigma))}{v^*(\theta_1-\theta_2+\theta_1\theta_2)\rho_1\sigma} & \rho_1 + \frac{\theta_2(1-\kappa\sigma)}{(\theta_1-\theta_2+\theta_1\theta_2)\sigma} \\ -\frac{(\theta_1-\theta_2)(\kappa+\rho_2)(1-\kappa\sigma)}{v^*(\theta_1-\theta_2+\theta_1\theta_2)\rho_2} & -\frac{\theta_2\rho_1(\kappa+\rho_2)(1-\kappa\sigma)}{(\theta_1-\theta_2+\theta_1\theta_2)\rho_2(\rho_1\sigma-(1-\kappa\sigma))} \end{bmatrix} \begin{bmatrix} \frac{v^*\theta_1\rho_2(1-\kappa\sigma)}{(\theta_1-\theta_2+\theta_1\theta_2)(\kappa+\rho_2)\sigma^2} \\ \frac{\theta_1\rho_2(1-\kappa\sigma)(\rho_1\sigma-(1-\kappa\sigma))}{(\theta_1-\theta_2+\theta_1\theta_2)\rho_1(\kappa+\rho_2)\sigma^2} \\ \rho_2 - \frac{\theta_1(1-\kappa\sigma)}{(\theta_1-\theta_2+\theta_1\theta_2)\sigma} \end{bmatrix}. \quad (24)$$

The characteristic polynomial of the Jacobian (24) are given by

$$|\Theta I - J^*| = \Theta^3 + c_1\Theta^2 + c_2\Theta + c_3 = 0, \quad (25)$$

where

$$c_1 \equiv -\text{tr } J^*, \\ = -(\rho_1 + \rho_2), \quad (26)$$

$$c_2 \equiv \text{the sum of all principal minors of order 2 of } J^*, \\ = \rho_1\rho_2\sigma + \frac{(1-\kappa\sigma)(\theta_1\rho_2 - \theta_2\rho_1)}{\theta_1 - \theta_2 + \theta_1\theta_2}, \quad (27)$$

$$c_3 \equiv -\det J^*, \\ = -\frac{(\theta_1 - \theta_2)\rho_1\rho_2(1-\kappa\sigma)}{\theta_1 - \theta_2 + \theta_1\theta_2}. \quad (28)$$

The main purpose of this section is to show the existence of cycles for the canonical system (17)–(19). We do this by applying the Hopf bifurcation theorem. According to Feichtinger and Novak [1], we make use of the following lemmas.

Lemma 3.1. A 3×3 matrix J has a pair of purely imaginary eigenvalues, if and only if $c_2 > 0$ and $c_1c_2 - c_3 = 0$.

Proof. See Lemma 4.1 of Feichtinger and Novak [1]. $\square$

Lemma 3.2. Suppose that a 3×3 matrix J depends on a parameter κ , such that J possesses a pair of complex conjugate eigenvalues $\Theta_{1,2} = \tau(\kappa) \pm i\omega(\kappa)$, which are purely imaginary at $\kappa = \kappa_0$, i.e., $\tau(\kappa_0) = 0$, $\omega(\kappa_0) \neq 0$. Then, the crossing velocity is given by

$$\left. \frac{\partial \tau(\kappa)}{\partial \kappa} \right|_{\kappa=\kappa_0} = \frac{1}{2[c_2 + (c_1)^2]} \left[\frac{\partial}{\partial \kappa} (c_3 - c_1c_2) \right]. \quad (29)$$

Proof. See Lemma 4.2 of Feichtinger and Novak [1]. $\square$

Proposition 3.1. The conditions of the Hopf bifurcation theorem (existence part) are fulfilled for $\kappa = \kappa_0$, if $\kappa > 1/\sigma - \rho_1$ and $\theta_1 > \theta_2$ hold.

Proof. (i) The conditions $c_2 > 0$ and $c_1 c_2 - c_3 = 0$ in Lemma 3.1. are equivalent to $c_3/c_1 = c_2 > 0$. Thus, in our case, the bifurcation condition is

$$(\rho_1 + \rho_2) \left[\rho_1 \rho_2 \sigma + \frac{(1 - \kappa \sigma)(\theta_1 \rho_2 - \theta_2 \rho_1)}{\theta_1 - \theta_2 + \theta_1 \theta_2} \right] = \frac{(\theta_1 - \theta_2) \rho_1 \rho_2 (1 - \kappa \sigma)}{\theta_1 - \theta_2 + \theta_1 \theta_2} > 0. \quad (30)$$

Choosing κ as bifurcation parameter, we get the following critical value

$$\kappa_0 = \frac{1}{\sigma} - \frac{(\theta_1 - \theta_2 + \theta_1 \theta_2) \rho_1 \rho_2 (\rho_1 + \rho_2)}{\theta_2 \rho_1^2 - \theta_1 \rho_2^2}. \quad (31)$$

As $\theta_1 > \theta_2$, the right-hand side of (30) is positive.

(ii) Applying (29) to calculate the crossing velocity, we obtain

$$\left. \frac{\partial \tau(\kappa)}{\partial \kappa} \right|_{\kappa=\kappa_0} = \frac{(\theta_2 \rho_1^2 - \theta_1 \rho_2^2)^2 / \{2(\theta_1 - \theta_2 + \theta_1 \theta_2)\}}{[(\theta_2 \rho_1^2 - \theta_1 \rho_2^2)(\rho_1 + \rho_2)^2 + (\theta_1 - \theta_2) \rho_1^2 \rho_2^2]}. \quad (32)$$

As $\kappa > 1/\sigma - \rho_1$ has to hold for the critical value (31), we have

$$\frac{\rho_1}{\rho_2} > \frac{\theta_1 - \theta_2 + \theta_1 \theta_2}{2\theta_2} + \frac{\sqrt{(\theta_1 - \theta_2 + \theta_1 \theta_2)^2 + 4\theta_2(2\theta_1 - \theta_2 + \theta_1 \theta_2)}}{2\theta_2}. \quad (33)$$

Moreover, the right-hand side of (33) is greater than 1, and we get the following inequality

$$\frac{\rho_1}{\rho_2} > \frac{\theta_1}{\theta_2} > 1. \quad (34)$$

Therefore, it is clear that the right-hand side of (32) is positive. $\square$

From (34), we may give the following interpretation. That is, a Hopf bifurcation can occur if workers are less farsighted than capitalists, and are wasteful (a penalty for the consumption rate which rises in the neighbourhood of the point 1). These conditions seem to be suitable from a practical point of view.

3.4. Numerical Example

We will present a numerical example leading to cycles. We choose the following parameter values:

$$\theta_1 = 15, \theta_2 = 3.3, \rho_1 = 0.71, \rho_2 = 0.02, \sigma = 1.5.$$

Using (31), we compute the critical value $\kappa_0 = 0.283929$. The steady state in this case is given by

$$(v^*, \lambda^*, \mu^*) = (0.963325, 0.808601, 21.2947).$$

As the crossing velocity (32) is positive, namely $\partial \tau(\kappa)/\partial \kappa = 0.025344$, cycles occur for $\kappa > \kappa_0$.

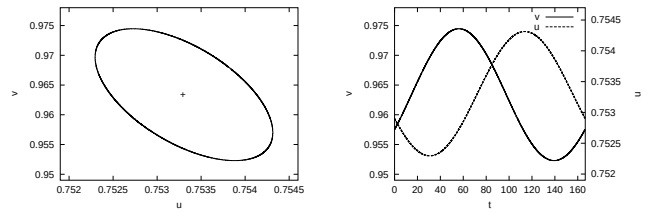


Figure 1: Cycle in the (u, v) plane and time paths of $u(t), v(t)$ for $\kappa = 0.28393$.

4. Conclusions

As we reduced Goodwin's two-state-variable system to an one-state-variable system for reasons of analytical convenience, our model is very simple and stylized. However, we presented that the existence of cyclical behaviour of workers' share of income, the employment rate and the consumption rate both analytically and numerically. This results accord with the spirit of Goodwin's model, namely, partly complementary, partly hostile relationship between workers and capitalists. More important, the fluctuations which our system generates are the result of rational behaviour of workers and capitalists.

References

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