

Synchronized states observed in coupled four oscillators

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1. Introduction

Systems of coupled oscillators are widely used as models for biological rhythmic oscillations such as human circadian rhythms[1, 2], finger movements, animal locomotion[3], swarms of fireflies that flash in synchrony, synchronous firing of cardiac pacemaker cells[5, 6], and so on. Using these coupled oscillator models, many investigators have studied the mechanism of generation of synchronous oscillation and phase transitions between distinct oscillatory modes. From the standpoint of bifurcation, the former and the latter correspond to the Hopf bifurcation of an equilibrium point (or the tangent bifurcation of a periodic solution) and the pitchfork bifurcation (or the period-doubling bifurcation) of a periodic solution, respectively. Using group theoretic discussion applied to the coupled oscillators, we can derive some general theorems concerning with the bifurcations of equilibrium points and periodic solutions[7].

In the study of coupled oscillator system, the four-coupled oscillator system is one of the most interesting system, because there exists an irregular degenerate oscillatory mode (or an independent pair of anti-phase mode) [8, 9] when the equation of the single oscillator is invariant under inversion of state variables.

Mishima and Kawakami studied the oscillatory modes generated by the Hopf bifurcations of the origin (equilibrium point) in several systems of coupled four BVP (Bonhöffer-van der Pol) oscillators [10]. However, they considered the Hopf bifurcation of the origin, because only it is supercritical. Tsumoto *et al.* investigated bifurcations of the Modified BVP (MBVP) equation[11]. In the MBVP system, the supercritical Hopf bifurcation of non-origin equilibrium points occurs.

In this paper, we examine the oscillatory modes generated by the Hopf bifurcations of non-origin equilibrium points in the four-coupled oscillator system. The Hopf bifurcations of the equilibrium points with strong symmetrical property and the generated oscillatory modes are classified. We ob-

serve three-phase, four-phase, in-phase and a pair of anti-phase synchronized states. The three-phase and four-phase solutions meet the Neimark-Sacker bifurcations, and three-phase and four-phase quasi-periodic solutions are generated, respectively.

2. Circuit Equation

We consider the coupled MBVP oscillator system shown in Fig. 1. The circuit equation is described as

$$\begin{aligned} L_1 \frac{di_{k1}}{dt} &= -R_1 i_{k1} - v_k \\ L_2 \frac{di_{k2}}{dt} &= -R_2 i_{k2} - v_k \\ C \frac{dv_k}{dt} &= i_{k1} + i_{k2} - g(v_k) \\ &\quad - G(3v_k - v_{k+1} - v_{k-1} - v_{k+2}) \\ (k &= 1, \dots, 4, v_0 \equiv v_4, v_5 \equiv v_1, v_6 \equiv v_2), \end{aligned} \quad (1)$$

where the nonlinear conductance $g(v_k)$ is assumed to be

$$g(v_k) = -v_k + \frac{1}{3}v_k^3. \quad (2)$$

The values of system parameters are fixed as [11]

$$L_1^{-1} = 0.2, R_1 = 4.0, R_2 = 2.1, C^{-1} = 3.0. \quad (3)$$

The Jacobi matrix of Eq. (1) is described by

$$DF = \begin{bmatrix} X_0 & X_1 & X_1 & X_1 \\ X_1 & X_0 & X_1 & X_1 \\ X_1 & X_1 & X_0 & X_1 \\ X_1 & X_1 & X_1 & X_0 \end{bmatrix}. \quad (4)$$

Each block is given by

$$X_0 = \begin{bmatrix} -R_1 L_1^{-1} & 0 & -L_1^{-1} \\ 0 & -R_2 L_2^{-1} & -L_2^{-1} \\ C^{-1} & C^{-1} & C^{-1}(1 - v_{*k}^2) - 3d \end{bmatrix},$$

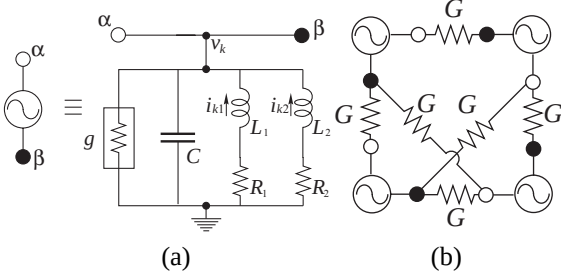


Figure 1: MBVP circuit (a) and coupled system (b).

$$X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d \end{bmatrix} \quad (5)$$

where v_{*k} is a coordinate of an equilibrium point and $d = C^{-1}G$. Using orthogonal matrix given by

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} I & I & O & 1/\sqrt{2} I \\ 1/\sqrt{2} I & O & I & -1/\sqrt{2} I \\ 1/\sqrt{2} I & -I & O & 1/\sqrt{2} I \\ 1/\sqrt{2} I & O & -I & -1/\sqrt{2} I \end{bmatrix} \quad (6)$$

we diagonalize the Jacobian matrix (4) as:

$$Q^{-1} \cdot DF \cdot Q = \begin{bmatrix} Y_0 & O & O & O \\ O & Y_1 & O & O \\ O & O & Y_1 & O \\ O & O & O & Y_1 \end{bmatrix} \quad (7)$$

where

$$Y_0 = X_0 + 3X_1, \quad (8)$$

$$Y_1 = X_0 - X_1, \quad (9)$$

$$(10)$$

In the next section we classify the oscillatory modes generated by the Hopf bifurcation in each block Y_l ($l = 1, 2$) in Eq. (7).

3. Results

Table 1: Classification of oscillatory modes.

equilibrium point	oscillatory modes
(a_1, a_1, a_1, a_1)	4-phase, in-phase(1)
$(a_2, a_2, -a_2, -a_2)$	a pair of anti-phase, a pair of in-phase
(a_3, a_3, a_3, b)	3-phase, in-phase(2)

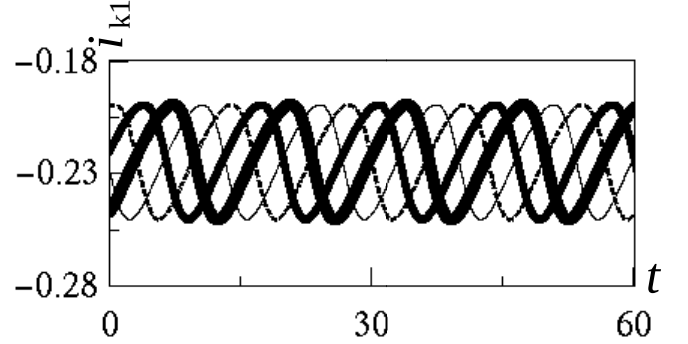


Figure 2: Waveforms of a 4-phase synchronized state. $L_2^{-1} = 0.035$. $d = -0.01$.

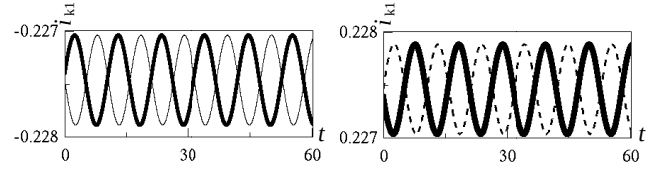


Figure 3: Waveforms of a pair of an anti-phase synchronized state. $L_2^{-1} = 0.048$. $d = -0.0016$.

In Eq. (1) there exists three equilibrium points satisfying $v_{*1}^2 = v_{*2}^2 = v_{*3}^2 = v_{*4}^2$; those are $(v_{*1}, v_{*2}, v_{*3}, v_{*4}) = (0, 0, 0, 0)$, (a_1, a_1, a_1, a_1) and $(a_2, a_2, -a_2, -a_2)$. Here we study the supercritical Hopf bifurcations of equilibrium points (a_1, a_1, a_1, a_1) and $(a_2, a_2, -a_2, -a_2)$. And also (a_3, a_3, a_3, b) type equilibrium point is studied, because this type has the supercritical Hopf bifurcations.

In Tab. 1 we classify oscillatory modes generated by the Hopf bifurcations of three types of equilibrium points. For each equilibrium point, an in-phase solution is generated by the Hopf bifurcation in the block Y_0 in Eq. (7). On the other hand, the Hopf bifurcations in the blocks Y_1 are degenerate (three pairs of eigenvalues satisfy the condition of the Hopf bifurcation). Thus many oscillatory modes (stable and unstable) appear simultaneously by the degenerate Hopf bifurcation.

Waveforms of four-phase, a pair of anti-phase, three-phase and in-phase(2) solutions are shown in Figs. 2, 3, 4 and 5, respectively. In each waveform, there is no correspondence between the kinds of curved line and the order of oscillators denoted by k , because the oscillators are fully connected and we can exchange any oscillators.

From Fig. 4 we can see that three oscillators synchronized at three-phase (Fig. 4(a)) and oscillation of one oscillator is almost stopped (Fig. 4(b)). On the other hand, in Fig. 5 three oscillators are synchronized at in-phase but their oscillations are almost stopped (Fig. 5(a)), and only one oscillator has

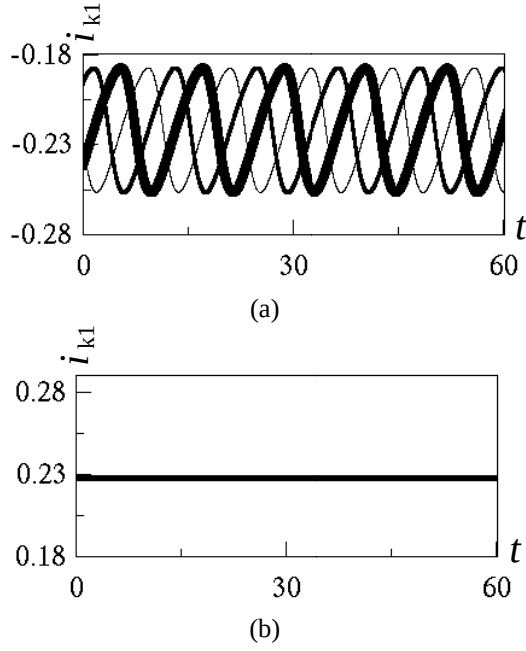


Figure 4: Waveforms of a 3-phase synchronized state. $L_2^{-1} = 0.06$. $d = -0.0016$.

large amplitude (Fig. 5(b)).

By changing the values of parameter L_2^{-1} and d , the three-phase and four-phase solutions meet the Neimark-Sacker bifurcations, and three-phase and four-phase quasi-periodic solution appear as shown in Fig. 6 and 7, respectively.

Figures 8 and 9 show some interesting oscillatory modes generated by the degenerate Hopf bifurcations as mentioned earlier. In Fig. 8 the oscillations of two oscillators completely stopped and other two oscillators are synchronized at anti-phase. In Fig. 9 three oscillators are synchronized in-phase with small amplitude, and one oscillator is synchronized at anti-phase against other three ones and has large amplitude.

4. Conclusion

We have studied the oscillatory modes generated by the Hopf bifurcations in coupled four oscillators. The Hopf bifurcations of three types of equilibrium points and the generated oscillatory modes are classified. Moreover, by numerical bifurcation analysis we observed various interesting synchronized states caused by the degenerate Hopf bifurcations.

Considering the associative memory model for storing patterns as oscillatory states[12], this system has the advantage of many oscillatory modes.

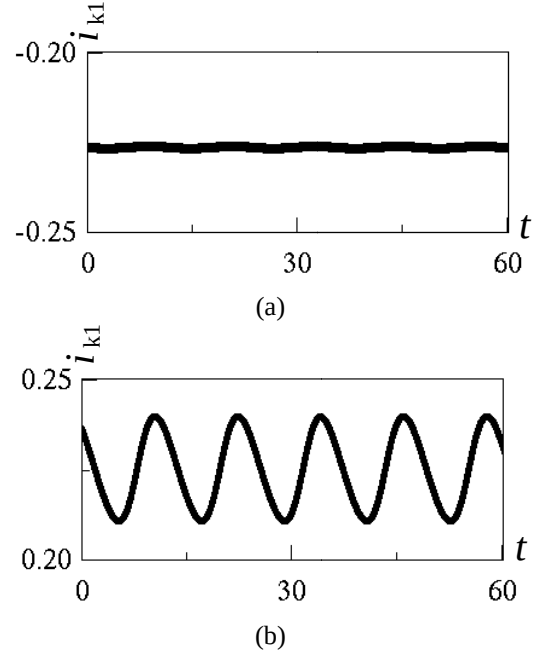


Figure 5: Waveforms of an in-phase synchronized state caused by the Hopf bifurcation of (a_3, a_3, a_3, b) . $L_2^{-1} = 0.06$. $d = 0.0005$.

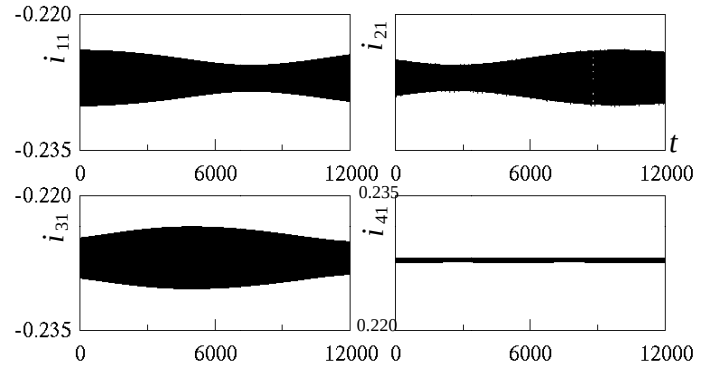


Figure 6: Waveforms. $L_2^{-1} = 0.04$. $d = -0.0015$.

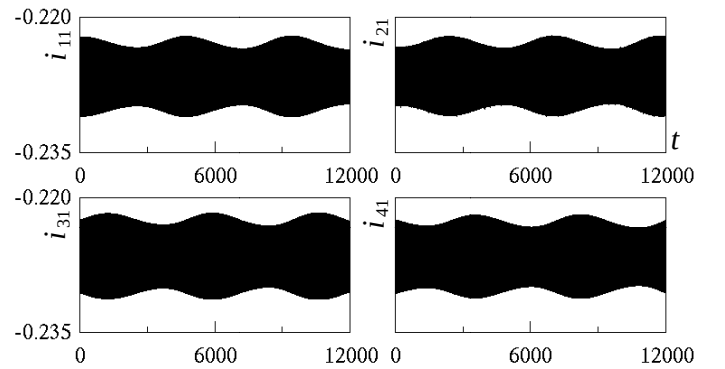


Figure 7: Waveforms. $L_2^{-1} = 0.06$. $d = -0.0021$.

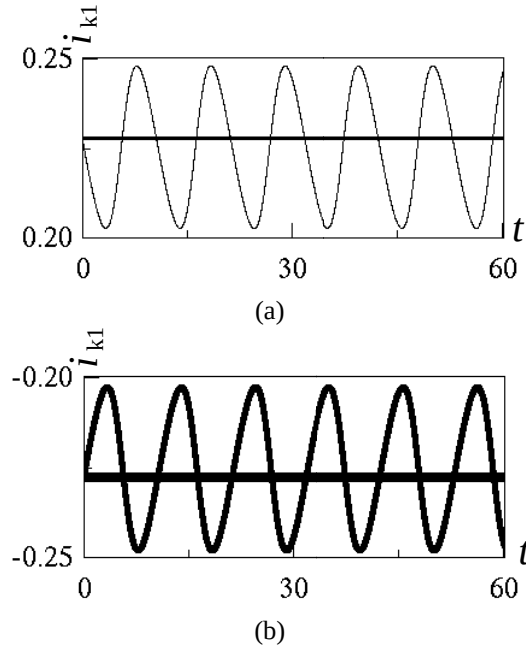


Figure 8: Waveforms of oscillation death (unstable). $L_2^{-1} = 0.06$. $d = -0.0023$.

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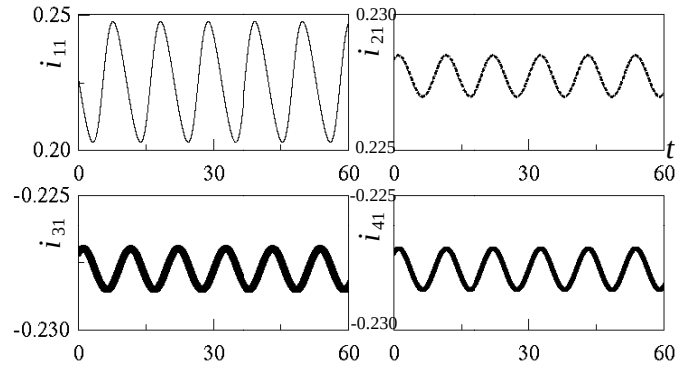


Figure 9: Waveforms of an interesting oscillatory mode (unstable). $L_2^{-1} = 0.06$. $d = -0.0021$.

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