Theorems on the Uniqueness of an Initial Solution for Homotopy Methods

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Abstract For the global convergence of homotopy methods, it is a necessary condition that an initial solution given is the unique solution to the homotopy equation. According to the conventional criterion, such an initial solution, however, is restricted in some very narrow region. In this paper, considering the circuit interpretation of homotopy equations, we study new criteria extending the region wherein any initial solution satisfies the uniqueness condition.

Keywords nonlinear circuit, circuit analysis, homotopy method, initial solution, unique solution

1. Introduction

In circuit simulation, finding DC operating points of nonlinear circuits is an important and difficult task. SPICE-type circuit simulators, widely used for designing LSI's, employ the Newton-Raphson (NR) method [1] for solving modified nodal (MN) equations [2]. However, the NR method or its variants often fail to converge to a solution unless the initial estimation point is close enough to the solution [1]. To overcome this convergence problem, globally convergent homotopy methods have been studied by many researchers from various viewpoints [3]–[11]. These studies are divided into two categories. The first one is how to construct a homotopy function and the second one is how to numerically trace a solution curve. Among those, the Newton fixed-point homotopy method and the spherical algorithm are well-known to be practical and efficient [5],[7], [9],[10]. In homotopy methods, the computational efficiency depends on the homotopy function and the curve tracing algorithm [3]. Clearly, it depends also on an initial solution given. However, there are few studies on this third category, how to set an initial solution[4].

For the global convergence of homotopy methods, it is a necessary condition that an initial solution given is the unique solution to the homotopy equation (the uniqueness condition)[3]. According to the conventional criterion, such an initial solution for some practical homotopy methods, however, is restricted in some very narrow region. In transistor circuits, for instance, it is restricted on a line in the branch voltage space of each BJT [4]. Note that no BJT on that line is in nomal (forward active) operation. (See Theorem 2 in Section 2.) Thus, we have to start at an initial solution very far from the normal operation.

In this paper, we study on the third category, the initial solution. Considering the circuit interpretation of homotopy equations, we show theorems extending the region wherein any initial solution satisfies the uniqueness condition. By these theorems, the region is extended into the entire space including the normal operation region. In the next section, as preliminaries, a homotopy method for MN equations and some related properties are reviewed. In Section 3, theorems extending the region are presented. In Section 4, numerical examples are shown.

2. Homotopy Method for MN Equations

We review a homotopy method for solving systems of equations of the form

$$\mathsf{f}(\mathsf{x}) = 0, \mathsf{f}(\cdot) : \mathsf{R}^n \to \mathsf{R}^n. \qquad (1)$$

In the MN equation, Eq. (1) is rewritten as follows [7],[8]:

$$\begin{aligned} \mathsf{f}_g(\mathsf{v}, \mathsf{i}) \quad &, \quad \mathsf{D}_g \mathsf{g}(\mathsf{D}_g{}^T \mathsf{v}) + \mathsf{D}_E \mathsf{i} + \mathsf{J} = 0 \\ \mathsf{f}_E(\mathsf{v}) \quad &, \quad \mathsf{D}_E{}^T \mathsf{v} - \mathsf{E} = 0 \end{aligned}$$

where $\mathsf{f} = (\mathsf{f}_g, \mathsf{f}_E)^T$, $\mathsf{f}_g : \mathsf{R}^n \to \mathsf{R}^N$, $\mathsf{f}_E : \mathsf{R}^N \to \mathsf{R}^M$, $\mathsf{x} = (\mathsf{v}, \mathsf{i}) \in \mathsf{R}^n$, and $n = N + M$. The variable vector $\mathsf{v} \in \mathsf{R}^N$ denotes the node voltages to the datum node and the variable vector $\mathsf{i} \in \mathsf{R}^M$ denotes the branch currents of the independent voltage sources. Also, the continuous function $\mathsf{g} : \mathsf{R}^K \to \mathsf{R}^K$ is a VCCS (voltage-controlled current source) type. In addition, D_g is an $N \times K$ reduced incidence matrix for the g branches and D_E is an $N \times M$ reduced incidence matrix for the independent voltage source branches. Moreover, $\mathsf{J} \in \mathsf{R}^N$ is the current vector of the independent current sources and $\mathsf{E} \in \mathsf{R}^{\mathsf{M}}$ is the voltage vector of the independent voltage sources.

In homotopy methods [3], we consider an auxiliary equation

$$\mathsf{f}^{\,0}(\mathsf{x}) = 0$$

with a known solution x^0 and construct a homotopy

$$\mathsf{h}(\mathsf{x}, t) = t\mathsf{f}(\mathsf{x}) + (1-t)\mathsf{f}^{\,0}(\mathsf{x})$$

where $\mathsf{h} : \mathsf{R}^{n+1} \to \mathsf{R}^n$ and $t \in [0,1]$ is the homotopy parameter. Then the solution curve of the homotopy equation

$$\mathsf{h}(\mathsf{x}, t) = 0$$

is traced from the known initial solution $(\mathsf{x}^0, 0)$ at $t = 0$. If the solution curve reaches the $t = 1$ hyperplane at $(\mathsf{x}^*, 1)$, then a solution x^* is obtained.

The Newton fixed-point homotopy method [10] utilizes the following auxiliary function:

$$\mathsf{f}^{\,0} = \mathsf{f}(\mathsf{x}) - \mathsf{f}(\mathsf{x}^0) + \mathsf{A}(\mathsf{x} - \mathsf{x}^0)$$

where A is some $n \times n$ matrix represented as follows:

$$\mathsf{A} = \begin{bmatrix} \mathsf{D_g}\mathsf{G_{FP}}\mathsf{D_g}^{\mathsf{T}} & 0 \\ 0 & -R_{\mathsf{FP}}\mathsf{1_M} \end{bmatrix}. \qquad (2)$$

In Eq. (2), $\mathsf{G_{FP}}$ =diag$(G_{\mathsf{FP}\,j})$, $j = 1, 2, \cdots, K$, is a positive semi-definite diagonal matrix, whose appropriate diagonal elements are positive and the others are zero. Also, R_{FP} is a scalar positive value and $\mathsf{1_M}$ is an $M \times M$ identity matrix. Then the homotopy is expressed as follows:

$$\begin{aligned} \mathsf{h}(\mathsf{x}, t) = \mathsf{f}(\mathsf{x}) &- (1-t)\mathsf{f}(\mathsf{x}^0) \\ &+ (1-t)\mathsf{A}(\mathsf{x} - \mathsf{x}^0). \end{aligned} \qquad (3)$$

Under some regularity assumptions of h, the solution curve of $\mathsf{h}(\mathsf{x}, t) = 0$ is guaranteed to reach the $t = 1$ hyperplane, if the uniqueness condition and the boundary free condition hold [3]. Many homotopy methods are proved to be globally convergent for the MN equations [7]–[10]. The uniqueness condition mentioned above is that an initail solution x^0 given is the unique solution to $\mathsf{h}(\mathsf{x}, t) = 0$. To discuss the condition, we use the following terminology [4].

Definition A continuous function $\mathsf{g} : \mathsf{R}^K \to \mathsf{R}^K$ is said to be strictly passive on $\mathsf{v_b^p}$ if there exists a $\mathsf{v_b^p} \in \mathsf{R}^K$ such that $(\mathsf{v_b} - \mathsf{v_b^p})^{\mathsf{T}}(\mathsf{g}(\mathsf{v_b}) - \mathsf{g}(\mathsf{v_b^p})) > 0$ holds for all $\mathsf{v_b} \in \mathsf{R}^K, \mathsf{v_b} \neq \mathsf{v_b^p}$. □

Under this passivity assumption, the following theorem holds [4],[7],[9], [10].

Theorem 1 In the homotopy method defined by Eq. (3), let $\mathsf{x}^0 = (\mathsf{v}^0, \mathsf{i}^0)$ be a solution to $\mathsf{h}(\mathsf{x}, 0) = 0$. Then x^0 is a unique solution if g is strictly passive on $\mathsf{v_b^0} = \mathsf{D_g}^{\mathsf{T}}\mathsf{v}^0$. □

Consider the strict passivity of transistor circuits. By using the Ebers-Moll model [12], the relation of the branch current vector $\mathsf{i_q}$ and the branch voltage vector $\mathsf{v_q}$ of a BJT is described as follows:

$$\mathsf{i_q}(\mathsf{v_q}) = \begin{bmatrix} i_{\mathsf{e}}(\mathsf{v_q}) \\ i_{\mathsf{c}}(\mathsf{v_q}) \end{bmatrix} = \mathsf{T}\,\mathsf{q}(\mathsf{v_q}) = \begin{bmatrix} q_{\mathsf{e}}(v_{\mathsf{e}}) - \alpha_{\mathsf{r}} q_{\mathsf{c}}(v_{\mathsf{c}}) \\ q_{\mathsf{c}}(v_{\mathsf{c}}) - \alpha_{\mathsf{f}} q_{\mathsf{e}}(v_{\mathsf{e}}) \end{bmatrix}$$

where

$$\begin{aligned} \mathsf{q}(\mathsf{v_q}) &= \begin{bmatrix} q_{\mathsf{e}}(v_{\mathsf{e}}) \\ q_{\mathsf{c}}(v_{\mathsf{c}}) \end{bmatrix} = \begin{bmatrix} m_{\mathsf{e}}(\exp(n_{\mathsf{e}}v_{\mathsf{e}}) - 1) \\ m_{\mathsf{c}}(\exp(n_{\mathsf{c}}v_{\mathsf{c}}) - 1) \end{bmatrix} \\ \mathsf{T} &= \begin{bmatrix} 1 & -\alpha_{\mathsf{r}} \\ -\alpha_{\mathsf{f}} & 1 \end{bmatrix}, \quad \mathsf{v_q} = \begin{bmatrix} v_{\mathsf{e}} \\ v_{\mathsf{c}} \end{bmatrix}. \end{aligned}$$

Also, i_{e} (i_{c}) denotes the emitter (collector) current and v_{e} (v_{c}) denotes the emitter (collector) voltage. The model parameters α, m and n are reqired to satisfy the passivity, no-gain and reciprocity conditions [11]. Moreover, $m_{\mathsf{e}}, n_{\mathsf{e}}, m_{\mathsf{c}}, n_{\mathsf{c}} > 0$ for pnp transistors and $m_{\mathsf{e}}, n_{\mathsf{e}}, m_{\mathsf{c}}, n_{\mathsf{c}} < 0$ for npn transistors.

As a conventional criterion for transistor circuits, the following theorem is used for the initial solution satisfying the uniqueness condition [4].

Theorem 2 A BJT described by the Ebers-Moll model is strictly passive on any $\mathsf{v_q^p}$ satisfying $n_{\mathsf{e}}v_{\mathsf{e}}^{\mathsf{p}} = n_{\mathsf{c}}v_{\mathsf{c}}^{\mathsf{p}}$. □

From Theorem 2, such a $\mathsf{v_q^p}$ is restricted on a line in the branch voltage space, where the emitter junction and the collector junction are both in the same state, both cutoff or both forward biased. Note that any forward active state does not satisfy Theorem 2.

3. Uniqueness Theorems

Many practical homotopy methods, such as the fixed-point homotopy method and the Newton fixed-point homotopy method, have their own circuit interpretations [9],[10]. When we use the homotopy (3), the equivalent circuit of the homotpy equation around a BJT is shown in Fig.1, which we call the compound transistor in this paper. In Fig.1, Q is an intrinsic transistor, r is an external resistance matrix, and G is a conductance matrix added by the above circuit interpretation, where

$$\mathsf{r} = \begin{bmatrix} r_{\mathsf{e}} + r_{\mathsf{b}} & r_{\mathsf{b}} \\ r_{\mathsf{b}} & r_{\mathsf{c}} + r_{\mathsf{b}} \end{bmatrix}, \quad \mathsf{G} = \begin{bmatrix} G_{\mathsf{E}} & 0 \\ 0 & G_{\mathsf{C}} \end{bmatrix}.$$

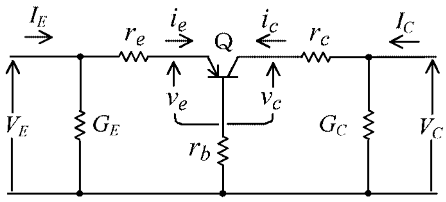

Fig.1 Compound transistor-1

The relation of the branch current vector I_b and the branch current vector V_b of the compound transistor-1 in Fig.1 are expressed as

$$\mathsf{I}_b = (I_{\mathsf{E}}, I_{\mathsf{C}})^{\mathsf{T}} = \mathsf{G}\mathsf{V}_b + \mathsf{i}_q$$
$$\mathsf{V}_b = (V_{\mathsf{E}}, V_{\mathsf{C}})^{\mathsf{T}} = \mathsf{v}_q + \mathsf{r}\,\mathsf{i}_q.$$

Such a compound transistor is expected to satisfy the strict passivity on a wider region than that of Theorem 2.

For the strict passivity of the compound transistor-1 in Fig.1, we can prove the following theorem.

Theorem 3 In the compound transistor-1, there exists a positive-definite diagonal matrix $\mathsf{G} = \mathsf{G}^0$ satisfying

$$G^0_{\mathsf{E}}(r_{\mathsf{e}} + r_{\mathsf{b}}(1-\alpha_{\mathsf{f}}))^2 + (r_{\mathsf{e}} + r_{\mathsf{b}}(1-\alpha_{\mathsf{f}})) - \frac{\alpha_{\mathsf{f}}^2}{4G^0_{\mathsf{C}}} \geq 0$$
$$G^0_{\mathsf{C}}(r_{\mathsf{c}} + r_{\mathsf{b}}(1-\alpha_{\mathsf{r}}))^2 + (r_{\mathsf{c}} + r_{\mathsf{b}}(1-\alpha_{\mathsf{r}})) - \frac{\alpha_{\mathsf{r}}^2}{4G^0_{\mathsf{E}}} \geq 0$$

such that the compound transistor-1 is strictly passive on any $\mathsf{v}^0_{\mathsf{q}} \in \mathsf{R}^2$. □

By Theorem 3, the region satisfying the uniqueness condition is extended into the entire space, if we set G an appropriate value. Thus, the uniqueness condition is satisfied for any initial solution given.

Next, we consider the case where the BJT model is simplified with $\mathsf{r} = 0$ and Theorem 3 cannot be applied. In such a case, we discuss the uniqueness condition under some restricted rigion $D1$. In transistor circuits, we can assume, from the no-gain property [11], such a $D1$ including all the possible solutions. If the strict passivity holds for all $\mathsf{v}_{\mathsf{b}} \in D1 \subset \mathsf{R}^{\mathsf{K}}$, $\mathsf{v}_{\mathsf{b}} \neq \mathsf{v}^{\mathsf{p}}_{\mathsf{b}}$, we call it strictly passive in $D1$. Then we have the following theorem.

Theorem 4 In the compound transistor-1 with $\mathsf{r} = 0$, suppose some region $D1 \subset \mathsf{R}^2$ where the upper limits of

$$\frac{\partial q_{\mathsf{e}}(v_{\mathsf{e}})}{\partial v_{\mathsf{e}}} \quad \text{and} \quad \frac{\partial q_{\mathsf{c}}(v_{\mathsf{c}})}{\partial v_{\mathsf{c}}}$$

are $q'_{\mathsf{e\ max}}$ and $q'_{\mathsf{c\ max}}$, respectively. Then there exists a matrix $\mathsf{G} = \mathsf{G}^0$ satisfying

$$G^0_{\mathsf{E}} \geq \alpha_{\mathsf{r}}^2 q'_{\mathsf{c\ max}}/4 \tag{4}$$
$$G^0_{\mathsf{C}} \geq \alpha_{\mathsf{f}}^2 q'_{\mathsf{e\ max}}/4 \tag{5}$$

such that in the region $D1$, the compound transistor-1 is strictly passive on any $\mathsf{v}^0_{\mathsf{q}} \in D1$. □

From Theorem 4, we can extend the region of the unique initial solution in the case of $\mathsf{r} = 0$.

In homotopy methods, we can also connect G in such a way as shown in Fig.3, which we call the compound transistor-2. For the passivity of the compound transistor-2 in Fig.3, we have the following theorems.

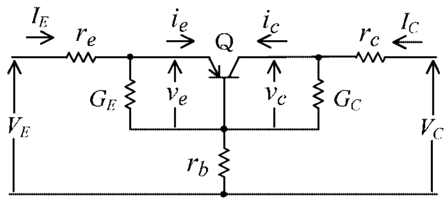

Fig.3 Compound transistor-2

Theorem 5 In the compound transistor-2, there exists a matrix $\mathsf{G} = \mathsf{G}^0$ satisfying

$$G^0_{\mathsf{C}} \geq \frac{\alpha_{\mathsf{f}}^2}{4(r_{\mathsf{e}} + r_{\mathsf{b}}(1-\alpha_{\mathsf{f}}))}$$
$$G^0_{\mathsf{E}} \geq \frac{\alpha_{\mathsf{r}}^2}{4(r_{\mathsf{c}} + r_{\mathsf{b}}(1-\alpha_{\mathsf{r}}))}$$

such that the compound transistor-2 is strictly passive on any $\mathsf{v}^0_{\mathsf{q}} \in \mathsf{R}^2$. □

From Theorem 5, we can extend the region of the unique initial solution for the compound transistor-2.

If we restrict $\mathsf{v}^0_{\mathsf{q}}$ within some desired region, we can relax the condition. The next theorem together with Theorem 4 is utilized in the subsequent theorem.

Theorem 6 In the compound transistor-2, there exists, for a positive-definite diagonal matrix $\mathsf{G} = \mathsf{G}^0$ arbitrary given, a $q'_{\mathsf{e\,min}}$ and a $q'_{\mathsf{c\,min}}$ satisfying

$$q'_{\mathsf{e\ min}} = \frac{1}{(1-\alpha_{\mathsf{r}}\alpha_{\mathsf{f}})}\left(\frac{\alpha_{\mathsf{r}}^2}{4(r_{\mathsf{c}} + r_{\mathsf{b}}(1-\alpha_{\mathsf{r}}))} - G^0_{\mathsf{E}}\right) \tag{6}$$
$$q'_{\mathsf{c\ min}} = \frac{1}{(1-\alpha_{\mathsf{r}}\alpha_{\mathsf{f}})}\left(\frac{\alpha_{\mathsf{f}}^2}{4(r_{\mathsf{e}} + r_{\mathsf{b}}(1-\alpha_{\mathsf{f}}))} - G^0_{\mathsf{C}}\right) \tag{7}$$

such that in $D2 = \{\mathsf{v}_{\mathsf{q}} = (v_{\mathsf{e}}, v_{\mathsf{c}})^{\mathsf{T}} | q'_{\mathsf{e}} \geq q'_{\mathsf{e\ min}}, q'_{\mathsf{c}} \geq q'_{\mathsf{c\ min}}\}$, the compound transistor-2 is strictly passive on any $\mathsf{v}^0_{\mathsf{q}} \in D2$. □

Theorem 7 In the compound transistor-2, let $D1$ and $D2$ be some regions where Theorems 4 and 6 hold, respectively. Then there exists a positive-definite diagonal matrix $\mathsf{G} = \mathsf{G}^0$ satisfying

$$\frac{1}{(1-\alpha_{\mathsf{r}}\alpha_{\mathsf{f}})}\Big(\frac{\alpha_{\mathsf{f}}^2}{4(r_{\mathsf{e}}+r_{\mathsf{b}}(1-\alpha_{\mathsf{f}}))} - G_{\mathsf{C}}^0\Big) - \frac{4G_{\mathsf{E}}^0}{\alpha_{\mathsf{r}}^2} \leq 0$$
$$\frac{1}{(1-\alpha_{\mathsf{r}}\alpha_{\mathsf{f}})}\Big(\frac{\alpha_{\mathsf{r}}^2}{4(r_{\mathsf{c}}+r_{\mathsf{b}}(1-\alpha_{\mathsf{r}}))} - G_{\mathsf{E}}^0\Big) - \frac{4G_{\mathsf{C}}^0}{\alpha_{\mathsf{f}}^2} \leq 0$$

such that $D = D_1 \cap D_2 \neq \varnothing$ holds and the compound transistor-2 is strictly passive on any $\mathsf{v}_{\mathsf{q}}^0 \in D$. □

In Theorem 7, if we assume the collector junction with non-forward biasing, then as shown in the next corollary we can extend the region of the unique initial solution with more relaxed condition.

Corollary In Theorem 7, there exist a $v_{\mathsf{e\ min}}$ satisfying Eq. (6), a G_{E}^0 satisfying

$$G_{\mathsf{E}}^0 = \frac{\alpha_{\mathsf{r}}^2}{4}G_{\mathsf{min}}, \quad G_{\mathsf{min}} = \frac{\partial q_{\mathsf{c}}}{\partial v_{\mathsf{c}}}\Big|_{\mathsf{v_c}=0}$$

and a G_{C}^0 satisfying Eqs. (5) and (7) such that for any $v_{\mathsf{e\ max}} > v_{\mathsf{e\ min}}$,

$$D = \left\{\mathsf{v}_{\mathsf{q}} = (v_{\mathsf{e}}, v_{\mathsf{c}})^{\mathsf{T}} \,|v_{\mathsf{c}} \leq 0,\ v_{\mathsf{e\ min}} \leq v_{\mathsf{e}} \leq v_{\mathsf{e\ max}}\right\}$$

hold. □

4. Numerical Exsamples

First, we confirm our theorems in a simple test circuit. We apply our thorems to a simple latch circuit known to possess three solutions [11]. Whenever connecting the conductances to each BJT as stated in the theorems, we can confirm a unique solution.

Next, we apply our theorems to the Newton fixed-point homtopy method for solving a DC operating point of the hybrid voltage-reference circuit [6],[10]. By our theorems, we can set an initial solution in the forward active region. Corollary is used to make the initial solution $\mathsf{v}_{\mathsf{q}}^0 = (0.7, 0)^{\mathsf{T}}$ unique. We use a typical set of model parameters, $r_{\mathsf{e}} = 10$, $r_{\mathsf{b}} = 100$, $r_{\mathsf{c}} = 300$. Then we have $G_{\mathsf{C}}^0 = 22.3 \times 10^{-3}$ and $G_{\mathsf{E}}^0 = 1.0 \times 10^{-12}$ satisfying Corollary. The number of steps for tracing the solution curve is 34, which is more efficient than that (68) of the conventional method with $\mathsf{v}_{\mathsf{q}}^0 = (0, 0)^{\mathsf{T}}$.

5. Conclusions

In this paper, we have presented theorems on the uniqueness of an initial solution for homotopy methods. We have also shown numerical examples to confirm the effectiveness of our theorems. These theorems extend the region wherein any initial solution guarantees the global convergence. By these theorems, the region is extended from a line into the entire space.

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