A New Approximate Analytic Method for Solving Nonlinear Vibration Problem

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Abstract

In this paper, the differential equation of the nonlinear vibration problem is firstly transformed into a nonlinear integral and differential equation. Then the solving problem of the equation is changed into a set of nonlinear algebraic equations with infinite unknown variables. According to the requested precision, the equation set can be truncated into a finite order nonlinear equation set when solved. Theoretically, the periodic solution with any accuracy can be obtained. As the results, Duffing free and force vibration system is discussed separately by using the method in this paper.

1. Introduction

The method to determine the periods of the periodic orbits of nonlinear dynamics systems is one of the most important fields in the nonlinear research, because it is related with many important problems such as bifurcation, stability problem and so on. There are many ways to determine the periodic orbit and its period of the nonlinear dynamics system. Those methods generally can be classified into two categories, namely, the frequency domain methods and the time domain methods. The former includes harmonic balancing and collocation methods, and the latter includes shooting method and Poincare method. In 1965, Urabe [1] used oscillation method for nonlinear periodic system. In 1966, Urabe [2] extended his investigation, and presented numerical computation of nonlinear force oscillations. In 1983, Lau and Chenung [3] gave an incremental harmonic balance method with multiple time scales for a periodic vibration of nonlinear system. In 2001, Zhou [4] research on the periodic orbit of nonlinear dynamic systems using Chebyshev polynomials and presented a new approach to solving a type of the vibration problem. In 1997, Sundararajan and Noah [5] presented a shooting algorithm with pseudo-arc length continuation for a nonlinear dynamic system subjected to periodic excitation. Classical perturbation method is only suitable to weaken nonlinear system [6]. For continuous method, in order to pass through a turning point that generally involves a stable and an unstable branch, a shooting algorithm with pseudo-arc length continuation was presented for nonlinear dynamic system subjected to periodic excitation in paper [5].

This work was supported by the National Science Foundation of the P. R. China (19990510).

In this paper, a new method is presented for solving nonlinear vibration problem. The differential equation of the nonlinear vibration problem is transformed into the nonlinear integral and differential equation. Then the problem of solving the equation is changed into that of the nonlinear algebraic equations with infinite unknown variables. Theoretically, the periodic solution with any accuracy can be obtained. As the results, Duffing free and force vibration system are discussed separately by using the method in this paper.

2. Method Analysis

Suppose a governing equation of a nonlinear vibration as follows,

$$D_1 x(t) + D_2 x(t) = P(t) \tag{1}$$

Where $D_1 x(1)$, $D_2 x(2)$ are the line part and the nonlinear part of the differential equation separately. In order to solve equation (1), firstly, consider the problem as follows:

$$D_1 x(t) = \delta(t - t_0)\,,\quad t, t_0 \in [0, T] \tag{2}$$

Where $\delta(t - t_0)$ is Delta function, T is a period of periodic solutions of the equation (1). Expand the equation (2) into Fourier series, and mark it $G(t, t_0)$, that is

$$G(t,t_0) = g_{10} + \sum_{n=1}^{\infty} g_{1n}\cos(n\omega t) + \sum_{n=1}^{\infty} g_{2n}\sin(n\omega t) \tag{3}$$

Where $\omega = 2\pi/T$, at the same time, expand $\delta(t-t_0)$ into Fourier series as follows.

$$\delta(t-t_0) = \frac{1}{T} + \sum_{n=1}^{\infty}\frac{2}{T}\cos(n\omega t_0)\cos(n\omega t) + \sum_{n=1}^{\infty}\frac{2}{T}\sin(n\omega t_0)\sin(n\omega t) \tag{4}$$

Substituting (3) and (4) into (2) separately, comparing the coefficients of the maturity series 1, $\cos(\omega t)$, $\sin(\omega t)$, $\cos(2\omega t)$, $\sin(2\omega t)$, …, we can worked out g_{1n}(n=0, 1, 2, , …) and g_{2n} (n=0, 1, 2, …). They are the functions of t_0. The equation (1) can be transformed into a nonlinear integral equation as follows by use of $G(t,t_0)$..

$$x(t) = -\int_0^T D_2 x(t_0) G(t,t_0)dt_0 + \int_0^T P(t_0)G(t,t_0)dt_0 \tag{5}$$

Substituting (3) into (5), we can get the x(t) as follows.

$$\begin{aligned} x(t) &= \int_0^T (P(t_0) - D_2 x(t_0)) g_{10} dt_0 \\ &+ \sum_{n=1}^{\infty}\int_0^T (P(t_0) - D_2 x(t_0)) g_{1n} dt_0 \cos(n\omega t) \\ &+ \sum_{n=1}^{\infty}\int_0^T (P(t_0) - D_2 x(t_0)) g_{2n} dt_0 \sin(n\omega t) \end{aligned} \tag{6}$$

Let

$$\left.\begin{aligned} x_{10} &= \int_0^T (P(t_0) - D_2 x(t_0)) g_{10} dt_0 \\ x_{1n} &= \int_0^T (P(t_0) - D_2 x(t_0)) g_{1n} dt_0 \\ x_{2n} &= \int_0^T (P(t_0) - D_2 x(t_0)) g_{2n} dt_0 \end{aligned}\right\} \text{(n=1, 2, 3, …)} \tag{7}$$

(6) is transformed into (8) as follows

$$x(t) = x_{10} + \sum_{n=1}^{\infty} x_{1n}\cos(n\omega t) + \sum_{n=1}^{\infty} x_{2n}\sin(n\omega t) \tag{8}$$

Substituting (8) into (7), we get (9) forms a nonlinear equation set with unknown variables x_{1n}(n=0, 1, 2, …), x_{2n}(n=0, 1, 2, …). (1) is the free vibration problem when P(t) equals 0, and ω is variable also, in this case, we can solve the problem of the equation by adding its numbers under the initial conditions. By solving this nonlinear algebra equation set, we can get the periodic solution of the differential Eq. (1).

$$\left\{\begin{aligned} x_{10} &= \int_0^T (P(t_0) - D_2(x_{10} \\ &+ \sum_{m=1}^{\infty}(x_{1m}\cos(m\omega t_0) + x_{2m}\sin(m\omega t_0)))) g_{10} dt_0 \\ x_{1n} &= \int_0^T (P(t_0) - D_2(x_{10} \\ &+ \sum_{m=1}^{\infty}(x_{1m}\cos(m\omega t_0) + x_{2m}\sin(m\omega t_0)))) g_{1m} dt_0 \\ x_{2n} &= \int_0^T (P(t_0) - D_2(x_{10} \\ &+ \sum_{m=1}^{\infty}(x_{1m}\cos(m\omega t_0) + x_{2m}\sin(m\omega t_0)))) g_{2m} dt_0 \end{aligned}\right.$$

$$\text{(n=1, 2, …)} \tag{9}$$

3. Solving of Free Vibration Problem of Duffing System

Consider Duffing system as follows:

$$\ddot{x}(t) + ax(t) + bx^3(t) = 0 \qquad (a>0, b>0),$$
$$x(0) = A \qquad \ddot{x}(0) = 0, t\in[0,T] \tag{10}$$

Firstly, solve the solution of the bellow equation.

$$\ddot{x} + ax = \delta(t-t_0)$$

Expand the equation into a cosine series by using the method mentioned above.

$$G(t,t_0) = \frac{1}{aT} + \sum_{n=1}^{\infty}\frac{2}{T(a-n^2\omega^2)}\cos(n\omega t_0)\cos(n\omega t)$$

And we can get a nonlinear integral equation as follows.

$$x(t) = \sum_{n=1}^{\infty} x_n \cos(nwt) + x_0 \tag{11}$$

Where $x_0 = -\dfrac{b}{aT}\displaystyle\int_0^T x^3(t_0)dt_0$

$$x_n = -\frac{2b}{T(a-n^2\omega^2)}\int_0^T x^3(t_0)\cos(n\omega t_0)dt_0 \quad (n=1,2,\ldots)$$

Suppose that $x^3(t_0) = \sum_{n=0}^{\infty}\tilde{x}_n\cos(n\omega t_0)$,

According to Fourier series formula of multiplication of two functions in reference [7], we can get

$$\begin{aligned}\tilde{x}_n &= \frac{x_0^2 x_n}{4} + \frac{x_n}{2}\sum_{m=1}^{\infty}x_m^2 + \frac{1}{4}\sum_{m=1}^{\infty}x_0 x_m(x_{m+n} + x_{m-n}) \\ &+ \frac{1}{4}\sum_{m=1}^{\infty}(x_{m+n} + x_{m-n})\sum_{p=1}^{\infty}x_p(x_{p+m} + x_{p-m})\end{aligned}$$

So we can get

$$x_0 = -\frac{b}{a}\left(\frac{x_0^3}{4} + \frac{x_0}{2}\sum_{m=1}^{\infty} x_m^2 + \frac{1}{2}\sum_{m=1}^{\infty} x_0 x_m^2 + \frac{1}{2}\sum_{m=1}^{\infty} x_m \sum_{p=1}^{\infty} x_p (x_{p+m} + x_{p-m})\right)$$

$$x_n = -\frac{b}{a-n^2\omega^2}\left[\frac{x_0^2 x_n}{4} + \frac{x_n}{2}\sum_{m=1}^{\infty} x_m^2 + \frac{1}{4}\sum_{m=1}^{\infty} x_0 x_m (x_{m+n} + x_{m-n}) + \frac{1}{4}\sum_{m=1}^{\infty}(x_{m+n} + x_{m-n})\sum_{p=1}^{\infty} x_p (x_{p+m} + x_{p-m})\right]$$

（n=1，2，⋯）

Only retaining the x_0 and x_1 in the equation set mentioned above and using the initial condition x(0)=A, we can obtain the solution as follows

$$\left.\begin{aligned} x_0 &= -\frac{b}{a}\left(\frac{x_0^3}{4} + \frac{3}{2}x_0 x_1^2\right) \\ x_1 &= \frac{b}{\omega^2 - a}\left(\frac{3}{4}x_0^2 x_1 + \frac{3}{4}x_0 x_1^3\right) \\ x_0 + x_1 &= A \end{aligned}\right\} \quad (12)$$

Solving (12), we can get $x_0 = 0$, $x_1 = A$, $\omega = a^{\frac{1}{2}}(1 + \frac{3b}{4a}A^2)^{\frac{1}{2}}$

If $\frac{3b}{4a}A^2 << 1$, then $\omega \approx a^{\frac{1}{2}}(1 + \frac{3b}{8a}A^2)$,The results mentioned above are the same as that of the first order approximate solution in reference [8].

4. Solving of Force Vibration Problem of Duffing System

Consider the governing equation of Duffing system as follows:

$$\ddot{x}(t) + h\dot{x} - ax + \beta x^3 = F\cos(\lambda t) \quad (13)$$

Where $h, \alpha, \beta > 0$, $T = 2\pi/\lambda$. In order to solve the periodic solution that satisfies $x(t) = x(t+\overline{T})$ ($\overline{T}$ is the period), firstly, we solve the linear problem as follows.

$$\ddot{x} + h\dot{x} - \alpha x = \delta(t - t_0), \quad t \in [0, \overline{T}] \quad (14)$$

We get $G(t,t_0) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\omega t) + b_n \sin(n\omega t))$, where

$\omega = 2\pi/\overline{T}$, $a_0 = -\frac{1}{\overline{T}\alpha}$, $a_n = \frac{2}{\overline{T}}(-c_n \cos(n\omega t_0) - d_n \sin(n\omega t_0))$

$b_n = \frac{2}{\overline{T}}(d_n \cos(n\omega t_0) - c_n \sin(n\omega t_0))$

Where $c_n = \frac{\alpha + n^2\omega^2}{h^2 n^2 \omega^2 + (\alpha + n^2\omega^2)^2}$ $d_n = \frac{hn\omega}{h^2 n^2\omega^2 + (\alpha + n^2\omega^2)^2}$

So, we can obtain the integral equation as follows.

$$x(t) = E_{11}\int_0^T x^3(t_0)dt_0 + E_{22} + \sum_{n=1}^{\infty}[(-\beta\int_0^T a_n x^3(t_0)dt_0 + G_{1n})\cos(n\omega t) + (-\beta\int_0^T x^3(t_0)b_n dt_0 + G_{2n})\sin(n\omega t)] \quad (15)$$

Where $E_{11} = -a_0\beta$, $E_{22} = \frac{Fa_0}{\lambda}\sin(\lambda\overline{T})$,

$G_{1n} = \frac{2F}{\overline{T}}(-c_n F_{1n} - d_n F_{2n})$, $G_{2n} = \frac{2F}{\overline{T}}(-d_n F_{1n} - c_n F_{2n})$

$$F_{1n} = \begin{cases} \frac{\overline{T}}{2} & \lambda = n\omega, \\ \frac{\sin(n\omega + \lambda)\overline{T}}{2(\lambda + n\omega)} + \frac{\sin(n\omega - \lambda)\overline{T}}{2(n\omega - \lambda)} & \lambda \neq n\omega \end{cases}$$

$$F_{2n} = \begin{cases} 0 & \lambda = n\omega, \\ -\frac{\cos(n\omega + \lambda)\overline{T} - 1}{2(\lambda + n\omega)} - \frac{\cos(n\omega - \lambda)\overline{T} - 1}{2(n\omega - \lambda)} & \lambda \neq n\omega \end{cases},$$

in Eq.(15), let
$$\begin{cases} P_{10} = 2(E_{22} + E_{11}\int_0^{\overline{T}} x^3(t_0)dt_0) \\ P_{1n} = -\beta\int_0^{\overline{T}} a_n x^3(t_0)dt_0 + G_{1n} \\ P_{2n} = -\beta\int_0^{\overline{T}} b_n x^3(t_0)dt_0 + G_{2n} \end{cases} \quad (n = 1, 2, \cdots)$$

We get $x(t) = \frac{P_{10}}{2} + \sum_{n=1}^{\infty}(P_{1n}\cos(n\omega t) + P_{2n}\sin(n\omega t))$,

$$x^2(t) = \frac{P_{10}}{2} + \sum_{n=1}^{\infty}(P_{1n}\cos(n\omega t) + P_{2n}\sin(n\omega t))\cdot$$

According to Fourier series formula of multiplication of two functions in reference [7], we can get

$$\overline{P}_{1n} = \frac{P_{10}P_{1n}}{2} + \frac{1}{2}\sum_{m=1}^{\infty}[P_{1m}(P_{1m+n} + P_{1m-n}) + P_{2m}(P_{2m+n} + P_{2m-n})]$$

(n=0，1，2，……)

$$\overline{P}_{2n} = \frac{P_{10}P_{2n}}{2} + \frac{1}{2}\sum_{m=1}^{\infty}[P_{1n}(P_{2m+n} - P_{2m-n}) - P_{2m}(P_{1m+n} - P_{1m-n})$$

(n=1，2，……)

When $m - n < 0$, then $P_{1m-n} = P_{1n-m}$, $P_{2m-n} = -P_{2n-m}$,let

$$x^3(t) = \frac{\overline{\overline{P}}_{10}}{2} + \sum_{n=1}^{\infty}(\overline{\overline{P}}_{1n}\cos n\omega t + \overline{\overline{P}}_{2n}\sin n\omega t)$$

As the same, we get

$$\overline{\overline{P}}_{1n} = \frac{\overline{P}_{10}P_{1n}}{2} + \frac{1}{2}\sum_{m=1}^{\infty}[\overline{P}_{1m}(P_{1m+n} + P_{1m-n} + \overline{P}_{2m}(P_{2m+n} + P_{2m-n})]$$

$(n=0，1，2，\cdots\cdots)$

$$\overline{\overline{P}}_{1n}=\frac{\overline{P}_{10}P_{2n}}{2}+\frac{1}{2}\sum_{m=1}^{\infty}[\overline{P}_{1m}(P_{2m+n}+P_{2m-n}-\overline{P}_{2m}(P_{1m+n}-P_{1m-n})]$$

$(n=1，2，\cdots\cdots)$

Then, we get
$$\left.\begin{aligned}P_{10}&=2E_{22}+E_{11}\overline{\overline{P}}_{10}\overline{T}\\P_{1n}&=G_{1n}+\beta c_n\overline{\overline{P}}_{1n}+\beta d_n\overline{\overline{P}}_{2n}\\P_{2n}&=G_{2n}-\beta d_n\overline{\overline{P}}_{1n}+\beta c_n\overline{\overline{P}}_{2n}\end{aligned}\right\}\qquad(16)$$

$(n=1，2，\cdots\cdots)$

The formulas in the equation set (16) form a infinitude order nonlinear equation set with variables P_{10}, P_{1n}, P_{2n}, (n=1，2，…). This equation set can be truncated into a finitude order nonlinear equation set when solved. If the precision requested isn't satisfied, we can add the numbers of the variables. We adopt $\alpha=0.5$，$\beta=0.5$，$h=0.1$ in the following calculation. The error r equals to the right of the equation (13) minus the left that is substituted the obtained solutions for.

When $F=0.17$, we get three solutions that its period is the same as that of the excitement force. Their coefficients (P_{1i}、P_{2i} (i=1，2，…，5)) changing with λ are shown in table.1 and table.2. They have the same forms as follows.

$$\begin{aligned}x(t)=&P_{11}\cos(\omega t)+P_{13}\cos(3\omega t)+P_{15}\cos(5\omega t)\\&+P_{21}\sin(\omega t)+P_{23}\sin(3\omega t)+P_{25}\sin(5\omega t)\end{aligned}\qquad(17)$$

But when $\lambda=0.4$, the solution suddenly changes, it changes into the forms as follows.

$$\begin{aligned}&1.033622\cos(\omega t)+1.767922\times10^{-2}\cos(2\omega t)+6.371151\times10^{-2}\\&\cos(3\omega t)-4.758755\times10^{-2}\cos(5\omega t)-1.508538\times10^{-2}\cos(7\omega t)\\&+5.975854\times10^{-1}\sin(\omega t)-1.3063055\times10^{-2}\sin(2\omega t)+3.2148536\\&\times10^{-2}\sin(3\omega t)+5.1674613\times10^{-2}\sin(5\omega t)+0(10^{-3})\end{aligned}$$

This is the symmetry disrepair phenomenon. Where $0(10^{-3})$ denotes the same order as 10^{-3}.

When $F=0.4$ and $\lambda=2.2$, selecting the former 15 variables in equation set (16), we can get the periodic solution as follows.

$$\begin{aligned}&0.99195167638595-1.0330269860569\times10^{-1}\cos(\omega t)\\&+4.2851816397160\times10^{-4}\cos(2\omega t)\\&+5.8888467184268\times10^{-3}\sin(\omega t)+0(10^{-5})\end{aligned}$$

The precision of the solutions is obvious. Besides this, we get a series of sub-harmonic solutions, such as $\omega/2$，$\omega/3$，…，$\omega/23$ of the system.

Table.1 The change of the cosine coefficients of the solution in form of (17) with λ

λ		P_{11}x10	P_{13}x10	P_{15}x10^2	ɣ
0.9	*	1.15720	-0.33299	0.0	$0(10^{-3})$
	* *	-2.04143	0.38793	-0.44790	$0(10^{-4})$
		-1.29564	0.0	0.0	$0(10^{-4})$
0.8	*	8.75153	-1.50953	0.41035	$0(10^{-4})$
	* *	-9.59448	1.22621	0.0	$0(10^{-4})$
		-1.49547	0.0	0.0	$0(10^{-4})$
0.6	*	11.66813	-0.57823	-2.36270	$0(10^{-4})$
	* *	-11.47918	0.16905	1.12717	$0(10^{-4})$
		-2.00204	0.0	0.0	$0(10^{-4})$

Table.2 The change of the sine coefficients of the solution in form of (17) with λ

λ		P_{21}x10	P_{23}x10	P_{25}x10^{-2}	ɣ
0.9	*	17.91605	-1.33238	0.0	$0(10^{-4})$
	* *	17.73238	-1.28353	0.0	$0(10^{-4})$
		0.08587	0.0	0.0	$0(10^{-4})$
0.8	*	14.50894	0.17929	-1.18340	$0(10^{-4})$
	* *	13.09093	0.33796	-0.93495	$0(10^{-4})$
		0.10449	0.0	0.0	$0(10^{-4})$
0.6	*	8.91957	1.90414	0.0	$0(10^{-4})$
	* *	6.88978	1.28295	0.0	$0(10^{-4})$
		0.14218	0.0	0.0	$0(10^{-4})$

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