A Simple Piecewise-Constant Oscillator Coupled by a Capacitor

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1. Introduction

The circuits which consists of linear elements and dependent switches are important because of the following reasons; 1) Such circuit exhibits some interesting nonlinear phenomena. 2) There are many engineering circuits with switching elements, for example, DC to DC converter, A/D converter and so on.

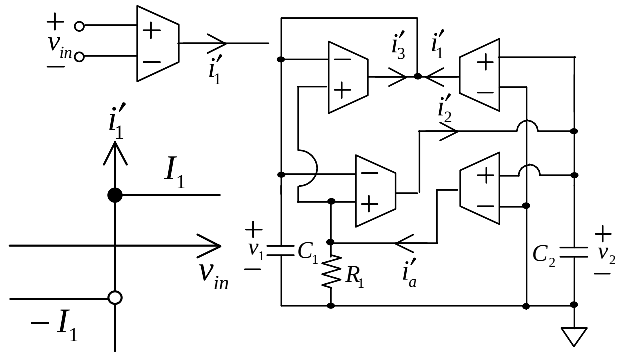
We have studied piecewise-constant systems (abbr. PWC) [1], [2]. PWCs are extremely simple switched dynamical systems, though exhibit some interesting phenomena. We have presented some theoretical results for bifurcation and chaos generation. However, theoretical analysis for higher dimensional PWC is hard, hence, it is important to analysis by using a computer-aided method.

On the other hand, coupled oscillator is useful model to describe various nonlinear phenomena in the field of natural science and to consider large scale nonlinear systems. A number of studies on coupled oscillators have been carried out ([3] and therein).

From these points of view, we propose a simple piecewise-constant oscillator coupled by a capacitor (abbr. PWOC) and show systematical method to analyze the PWOC. This circuit is simple, but exhibits interesting synchronization phenomena, chaotic phenomena and their co-existence. The circuit consists of linear capacitors, linear resistors and nonlinear voltage controlled current sources (abbr. VCCSs), where nonlinear VCCSs have Signam-like characteristics as comparators. The circuit dynamics is easily described by a explicit piecewise constant differential equation. In order to analyze the PWOC, we introduce hybrid return map, and show an algorithm to obtain rigorous solutions systematically. Using the algorithm, we can analyze the bifurcation phenomena which can be observed in the system.

2. Circuit Model

2.1. A Simple Piecewise-Constant Oscillator



(a) (b)

Figure 1: (a)Nonlinear VCCS. (b)Piecewise-constant oscillator. ($i_1' = I_1 \operatorname{sgn}(v_2)$, $i_2' = I_2 \operatorname{sgn}(I_a R_1 \operatorname{sgn}(v_2) - v_1)$, $i_3' = I_3 \operatorname{sgn}(I_a R_1 \operatorname{sgn}(v_2) - v_1)$, $i_a' = I_a \operatorname{sgn}(v_2)$).

In order to construct a simple piecewise-constant oscillator, we introduce a nonlinear voltage controlled current sources (abbr. VCCSs), which have Signam-like characteristics as comparators in Fig. 1(a):

$$\operatorname{sgn}(z) = \begin{cases} 1, & \text{for } z \geq 0, \\ -1, & \text{for } z < 0. \end{cases} \tag{1}$$

The VCCS can be realized by using operational transconductance amplifier (abbr. OTA). The circuit construction is shown in Fig. 1(b). The circuit consists of linear capacitors, linear resistors and VCCSs. The circuit dynamics are described as following:

$$\begin{cases} C_1 \dfrac{dv_1}{dt} = I_3 \operatorname{sgn}(I_a R_1 \operatorname{sgn}(v_2) - v_1) + I_1 \operatorname{sgn}(v_2), \\ C_2 \dfrac{dv_2}{dt} = I_2 \operatorname{sgn}(I_a R_1 \operatorname{sgn}(v_2) - v_1), \end{cases} \tag{2}$$

where I_1,I_2,I_3, and ,I_a are positive constant corresponding to saturation current of OTAs. Here, the following dimensionless variables and parameters are used:

$$\tau = \frac{I_1}{C_1 I_a R_1} t, x = \frac{1}{I_a R_1} v_1, y = \frac{C_2 I_1}{C_1 I_2 I_a R_1} v_2, \alpha = \frac{I_3}{I_1}. \tag{3}$$

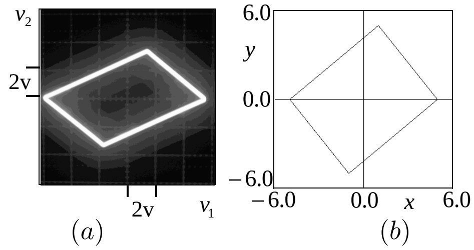


Figure 2: A periodic attractor. (a) Experimental result. (b) Computer calculated result. (Experimental: $R_a = 1$[kΩ], $C_1 = 10.0$[nF], $C_2 = 20.0$[nF], $I_1 = 1.0$[mA], $I_2 = 1.0$[mA], $I_3 = 0.2$[mA], $I_a = 3.0$[mA]. Computer calculation: $\alpha = 0.2$).

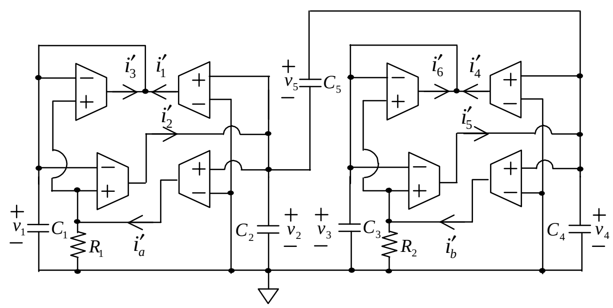


Figure 3: Piecewise-constant oscillator coupled by a capacitor. ($i_1' = I_1 \operatorname{sgn}(v_2)$, $i_2' = I_2 \operatorname{sgn}(I_a R_1 \operatorname{sgn}(v_2) - v_1)$, $i_3' = I_3 \operatorname{sgn}(I_a R_1 \operatorname{sgn}(v_2) - v_1)$, $i_a' = I_a \operatorname{sgn}(v_2)$, $i_4' = I_4 \operatorname{sgn}(v_4)$, $i_5' = I_5 \operatorname{sgn}(I_b R_2 \operatorname{sgn}(v_4) - v_3)$, $i_6' = I_6 \operatorname{sgn}(I_b R_2 \operatorname{sgn}(v_4) - v_3)$,$i_b' = I_b \operatorname{sgn}(v_4)$).

Then, equation (2) is normalized as

$$\begin{cases} \dot{x} = \alpha \cdot \operatorname{sgn}(\operatorname{sgn}(y) - x) + \operatorname{sgn}(y), \\ \dot{y} = \operatorname{sgn}(\operatorname{sgn}(y) - x), \end{cases} \tag{4}$$

where "$\cdot$" denotes differentiation by normalized time τ. Equation (4) has one parameter, for simplicity, we focus on the following parameter range:

$$0 < \alpha < 1. \tag{5}$$

In this parameter range, it is guaranteed that the system behaves without sliding mode and the trajectory convergences to a limit cycle for any initial condition. Fig. 2 shows experimental result and computer calculated result. The limit cycle of Fig. 2(b) is parallelogram of $(\frac{-1}{a}, 0)$, $(1, \frac{1}{a})$, $(\frac{1}{a}, 0)$, and, $(-1, \frac{-1}{a})$.

2.2. A Simple Piecewise-Constant Oscillator Coupled by a Capacitor

A simple piecewise-constant oscillator coupled by a capacitor (abbr. PWOC) is shown in Fig. 3. This circuit is two oscillators in Sec.2.1 coupled by a capacitor C_5. The circuit dynamics are described as following:

$$\begin{cases} C_1 \dfrac{dv_1}{dt} = I_3 \operatorname{sgn}(I_a R_1 \operatorname{sgn}(v_2) - v_1) + I_1 \operatorname{sgn}(v_2), \\ C_2 \dfrac{dv_2}{dt} = I_2 \operatorname{sgn}(I_a R_1 \operatorname{sgn}(v_2) - v_1) - C_5 \dfrac{d}{dt}(v_2 - v_4), \\ C_3 \dfrac{dv_3}{dt} = I_6 \operatorname{sgn}(I_b R_2 \operatorname{sgn}(v_4) - v_3) + I_4 \operatorname{sgn}(v_4), \\ C_4 \dfrac{dv_4}{dt} = I_5 \operatorname{sgn}(I_b R_2 \operatorname{sgn}(v_4) - v_3) + C_5 \dfrac{d}{dt}(v_2 - v_4). \end{cases} \tag{6}$$

where $I_1 \sim I_6$,I_a, and ,I_b are positive constant corresponding to saturation current of OTAs. Here, we introduce following dimensionless variables and parameters:

$$\begin{aligned} &\tau = \frac{I_1}{C_1 I_a R_1} t, x_1 = \frac{1}{I_a R_1} v_1, y_1 = \frac{C_0 I_1}{C_1 I_2 A_4 I_a R_1} v_2, \\ &x_2 = \frac{C_3 I_1}{C_1 I_4 I_a R_1} v_3, y_2 = \frac{C_0 I_1}{C_1 I_5 A_2 I_a R_1} v_4, \alpha = \frac{I_3}{I_1}, \\ &\beta = \frac{I_6}{I_4}, \gamma = \frac{C_5 I_5}{(C_4 + C_5) I_2}, \eta = \frac{C_5 I_2}{(C_2 + C_5) I_5}, \\ &\theta = \frac{C_1 I_4 I_a R_1}{C_3 I_1 I_b R_2}, C_0^2 = (C_2 + C_5)(C_4 + C_5) - C_5^2, \\ &A_2 = \frac{1}{C_0}(C_2 + C_5), A_4 = \frac{1}{C_0}(C_4 + C_5), A_5 = \frac{C_5}{C_0}. \end{aligned} \tag{7}$$

Equation (6) is normalized as

$$\begin{cases} \dot{x_1} = \alpha \operatorname{sgn}(\operatorname{sgn}(y_1) - x_1) + \operatorname{sgn}(y_1), \\ \dot{y_1} = \operatorname{sgn}(\operatorname{sgn}(y_1) - x_1) + \gamma \operatorname{sgn}(\operatorname{sgn}(y_2) - \theta x_2), \\ \dot{x_2} = \beta \operatorname{sgn}(\operatorname{sgn}(y_2) - \theta x_2) + \operatorname{sgn}(y_2), \\ \dot{y_2} = \eta \operatorname{sgn}(\operatorname{sgn}(y_1) - x_1) + \operatorname{sgn}(\operatorname{sgn}(y_2) - \theta x_2). \end{cases} \tag{8}$$

Equation (8) has five parameters, for simplicity, we focus on the following parameter range:

$$0 < \alpha < 1, 0 < \beta < 1, 0 < \gamma < 1, 0 < \eta < 1. \tag{9}$$

In this parameter range, the system behaves without sliding mode.

Piecewise-constant vector field of equation (8) consists of sixteen constant vector fields $\boldsymbol{a}(0) \sim \boldsymbol{a}(15)$. And let domains of the each constant vector fields be $D(0) \sim D(15)$. The local dynamics for $\boldsymbol{x} \in D(k)$ is given by $\dot{\boldsymbol{x}} \in \boldsymbol{a}(k)$. Here, in order to describe $\boldsymbol{a}(k)$ and $D(k)$, we introduce a dummy variable i:

$$\begin{aligned} &i \in \{0, 1, 2, \cdots, 15\}, \\ &i = \frac{-b(3) + 1}{2} \cdot 2^3 + \frac{-b(2) + 1}{2} \cdot 2^2 \\ &\quad + \frac{-b(1) + 1}{2} \cdot 2^1 + \frac{-b(0) + 1}{2} \cdot 2^0, \end{aligned} \tag{10}$$

where $b(0)$, $b(1)$, $b(2)$, and ,$b(3)$ are defined as following:

$$\begin{aligned} &b(3) = \operatorname{sgn}(\operatorname{sgn}(y_1) - x_1), b(2) = \operatorname{sgn}(y_1), \\ &b(1) = \operatorname{sgn}(\operatorname{sgn}(y_2) - \theta x_2), b(0) = \operatorname{sgn}(y_2). \end{aligned} \tag{11}$$

$D(k)$, $\boldsymbol{a}(k)$ are given by

$$\begin{aligned} D(k) &= \{(\boldsymbol{x}, i) | i = k\}, (k = 0, 1, 2, \cdots, 15) \\ \boldsymbol{a}(k) &= \{(\boldsymbol{x}, i) | i = k\} \end{aligned} \tag{12}$$

Note that $\boldsymbol{a}(k)$ is determined by the right-hand side of equation (8).

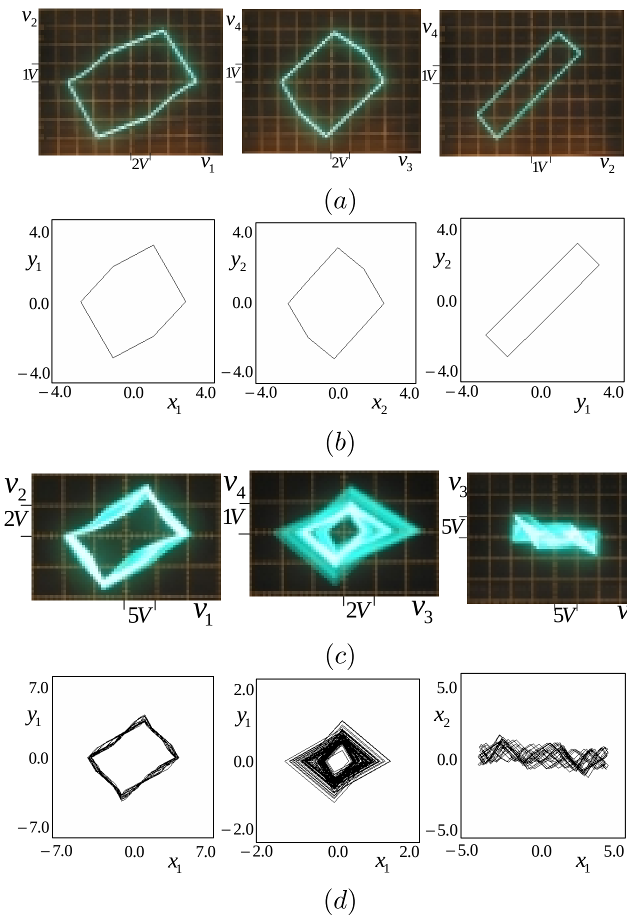


Figure 4: Typical attractors. (a), (b) Periodic. (c), (d) Chaotic.((a), (c): Laboratory measurements, (b), (d): Computer calculation), ($R_1 = 1$[kΩ], $R_2 = 1$[kΩ], $C_1 = 10.0$[nF], $C_2 = 20.0$[nF], $C_3 = 10.0$[nF], $C_4 = 20.0$[nF], $C_5 = 9.8$[nF], $I_1 = 9.0$[mA], $I_2 = 10.0$[mA], $I_3 = 2.2$[mA], $I_4 = 9.0$[mA], $I_5 = 10.0$[mA], $I_6 = 1.8$[mA], $I_a = 10.0$[mA], $I_b = 1.0$[mA], $\alpha = 0.24$, $\beta = 0.20$, $\gamma = 0.33$, $\eta = 0.33$, $\theta = 10.0$.)

3. Analysis

3.1. Calculate Rigorous Solutions

We can find the algorithm to calculate rigorous solutions. As shown in equation (8), the system has piecewise-constant vector field. The fact enable us to perform it.

First, we introduce hybrid return map [4] to obtain the rigorous solutions of equation (8). Let $\boldsymbol{A}$ be a set of all switching thresholds, let $\boldsymbol{x}_n \in \boldsymbol{R}^4$ be a real state vector on $\boldsymbol{A}$, and let $\boldsymbol{b}_n \in \boldsymbol{B}^4$ be a dummy binary state vector defined by $\boldsymbol{b}_n = (b_n(3), b_n(2), b_n(1), b_n(0))^t$. Then, we can define a return map $\boldsymbol{F}$ from $\boldsymbol{A}$ to itself:

$$\begin{aligned} &\boldsymbol{F} : \boldsymbol{R}^4 \times \boldsymbol{B}^4 \to \boldsymbol{R}^4 \times \boldsymbol{B}^4, \\ &(\boldsymbol{x}_{n+1}, \boldsymbol{b}_{n+1}) = \boldsymbol{F}(\boldsymbol{x}_n, \boldsymbol{b}_n). \end{aligned} \tag{13}$$

Second, based on the hybrid return map, we provide an algorithm to obtain rigorous solutions [5]. We show the algorithm to calculate rigorous solutions as following four steps:

STEP1: Let an initial value $\boldsymbol{x}_0$ be substituted for equation (11) and let these values be substituted for following equation:

$$\begin{aligned} i_0 = &\frac{-b_0(3)+1}{2} \cdot 2^3 + \frac{-b_0(2)+1}{2} \cdot 2^2 \\ &+ \frac{-b_0(1)+1}{2} \cdot 2^1 + \frac{-b_0(0)+1}{2} \cdot 2^0. \end{aligned} \tag{14}$$

We obtain the i_0 and $D(i_0)$ where $\boldsymbol{x}_0$ exists.

STEP2: We calculate the time t until $\boldsymbol{x}_0$ arrive at the border of $D(i_0)$. Let $\boldsymbol{E}(i)$ be a border of $D(i)$, let $\boldsymbol{E}'(i) \supset \boldsymbol{E}(i)$, and let $\boldsymbol{E}'(i)$ be represented by $\boldsymbol{E}'(i) = \{(\boldsymbol{x}, i) \mid x_1 = E_{x1}(i), y_1 = E_{y1}(i), x_2 = E_{x2}(i), y_2 = E_{y2}(i)\}$. Assuming the trajectory started from $\boldsymbol{x}_0$ hits $E_{y2}(i)$, $E_{x2}(i)$, $E_{y1}(i)$, and, $E_{x1}(i)$, each arrival times t_0, t_1, t_2, and, t_3 are given by

$$\begin{aligned} t_0 &= \frac{E_{y2}(i_0) - y_{02}}{a_{y2}(i_0)}, t_1 = \frac{E_{x2}(i_0) - x_{02}}{a_{x2}(i_0)}, \\ t_2 &= \frac{E_{y1}(i_0) - y_{01}}{a_{y1}(i_0)}, t_3 = \frac{E_{x1}(i_0) - x_{01}}{a_{x1}(i_0)}, \end{aligned} \tag{15}$$

respectively, where $ax_1(i)$, $ay_1(i)$, $ax_2(i)$, and $ay_2(i)$ are component of $\boldsymbol{a}(i)$: $\boldsymbol{a}(i) = (a_{x1}(i), a_{y1}(i), a_{x2}(i), a_{y2}(i))^t$. Then practical arrival time t is given by the minimum t_g omitting zero and negative.

$$t = \min_{g} \quad \{t_i\} \quad \text{for} \quad t_g > 0 \quad (g = 0, 1, 2, 3). \tag{16}$$

If all $t_g < 0$, the trajectory must not hit either borders, i.e., the trajectory diverges. But this situation never happen in parameter range (9).

STEP3: We calculate $b_1(0)$, $b_1(1)$, $b_1(2)$, $b_1(3)$, and ,$\boldsymbol{x}$. $\boldsymbol{x}_1$ is given by

$$\boldsymbol{x}_1 = \boldsymbol{x}_0 + \boldsymbol{a}(i_0) \cdot t. \tag{17}$$

If $t = t_j$, $b_1(j) = -b_0(j)$, where $j \in \{0, 1, 2, 3\}$. If $m \neq j$, $b_1(m)$, $(m = 0, 1, 2, 3)$, is given by substituting $\boldsymbol{x}_1$ into equation (11). i_1 are given by

$$\begin{aligned} i_1 = &\frac{-b_1(3)+1}{2} \cdot 2^3 + \frac{-b_1(2)+1}{2} \cdot 2^2 \\ &+ \frac{-b_1(1)+1}{2} \cdot 2^1 + \frac{-b_1(0)+1}{2} \cdot 2^0. \end{aligned} \tag{18}$$

STEP4: Let i_1, $\boldsymbol{x}_1$ be replaced with i_0, $\boldsymbol{x}_0$. Return to STEP2.

Fig. 4 shows experimental results and computer calculated rigorous solutions of the PWOC. Fig. 5 is shows a bifurcation diagram which is calculated by using this algorithm. Here, we introduce a definition for periodic attractor.

Definition: $\boldsymbol{\xi}$ is said to be a N-periodic point if $F^N(\boldsymbol{\xi}) = \boldsymbol{\xi}$ and $F^l(\boldsymbol{\xi}) \neq \boldsymbol{\xi}$ for $0 < l < N$, where F^N denotes the N-fold composition of F.

Fig. 5(a) and (b) represent a number of period N by gray scale: the depth of gray is in proportional to N. Fig. 5(c) is a summary of bifurcation diagram which shows co-existence regions. Fig. 4(a), (b) are observed in a parameter region "1" in Fig. 5(c) . Fig. 4(c), (d) are observed in a parameter region "10" in Fig. 5(c). They are observed on same parameter, namely, it is co-existence. We calculate the first $1-D$ Lyapunov exponent λ_1 for the case of Fig. 4(d). Because $\lambda_1 \cong 0.065$ is positive, chaos generation is confirmed .

4. Conclusions

In this paper, we have considered PWOC. Using the hybrid return map, we have derived the algorithm systematically to obtain rigorous solutions. Using the algorithm, we have derived bifurcation diagram for synchronous phenomena which can be observed in the system. Some analytical results are confirmed in the laboratory. Finally we enumerate future problems:
1) Detailed analysis for PWOC.
2) Analysis of large scale coupled oscillators of PWC .

References

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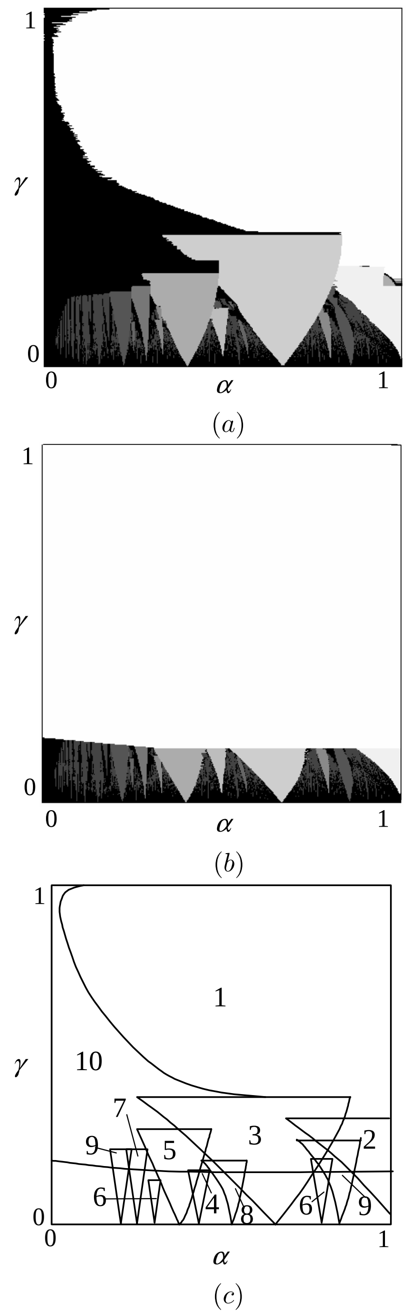

Figure 5: Bifurcation diagram ($\beta = 0.2$, $\eta = \gamma$, $\theta = 10.0$)(a) by increasing γ with fixed α. (b) by decreasing α with fixed γ. (c)Summary of bifurcation diagram(1: $N = 8$, 2: $N = 12$, 3: $N = 16$, 4: $N = 20$, 5: $N = 24$, 6: $N = 28$, 7: $N = 32$, 8: N= 36, 9: $N = 40$, 10: $N > 40$).