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Effect of Threshold in Symbolic Dynamics of a Chaotic Neuron Model

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1. Introduction

A chaotic neuron changes its internal state according to a certain bimodal map [1]. When it communicates with other neurons, its internal state is transformed into one of two separate outputs because of waveform shaping with a threshold on the axons. That is, if the value of the internal state is over a certain threshold, the output of the neuron is an excited state with firing, otherwise it is a resting state without firing. Figure 1 shows the graph of the chaotic neuron model with a typical set of the parameter values.

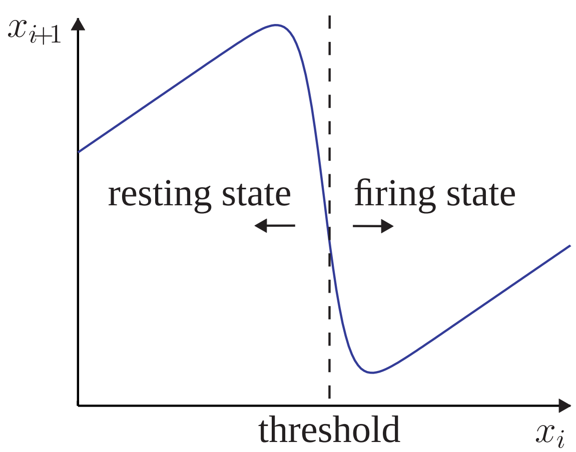


Figure 1: Bimodal map of a chaotic neuron

Though the two-value output of a chaotic neuron is directly dependent on the threshold level, it has not been clear what occurs in the output characteristics of the neuron when the threshold level is changed. In this work, we investigate the effect of the threshold. Specially, we address how the topological entropy, one of the most important dynamical invariants, behaves as the threshold level is changed.

2. Symbolic dynamics and topological entropy

We begin by studying the following piecewise linear bimodal map with an initial condition $x_0 = x$:

$$x_{i+1} = f(x_i) = \begin{cases} 4ax_i + 1 - a & \text{if } x_i < 1/4 \\ -2x_i + 3/2 & \text{if } x_i < 1/2 \\ 4a(x_i - 1) + a & \text{if } x_i < 1/2 \end{cases}, \quad (1)$$

for $i = 0, 1, 2\ldots$ where $0 < a \leq 1$ is a parameter. The output y_i of this system is obtained as follows:

$$y_i = \mathcal{H}(x_i - \theta)\,, \quad (2)$$

where $\mathcal{H}(x)$ is a Heaviside's step function, and θ is a threshold, such that $1/4 \leq \theta \leq 3/4$ (Figure 2).

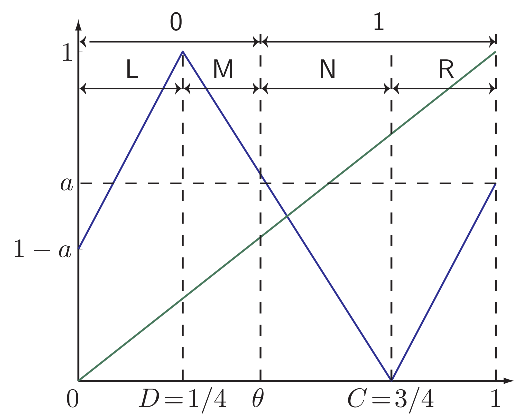
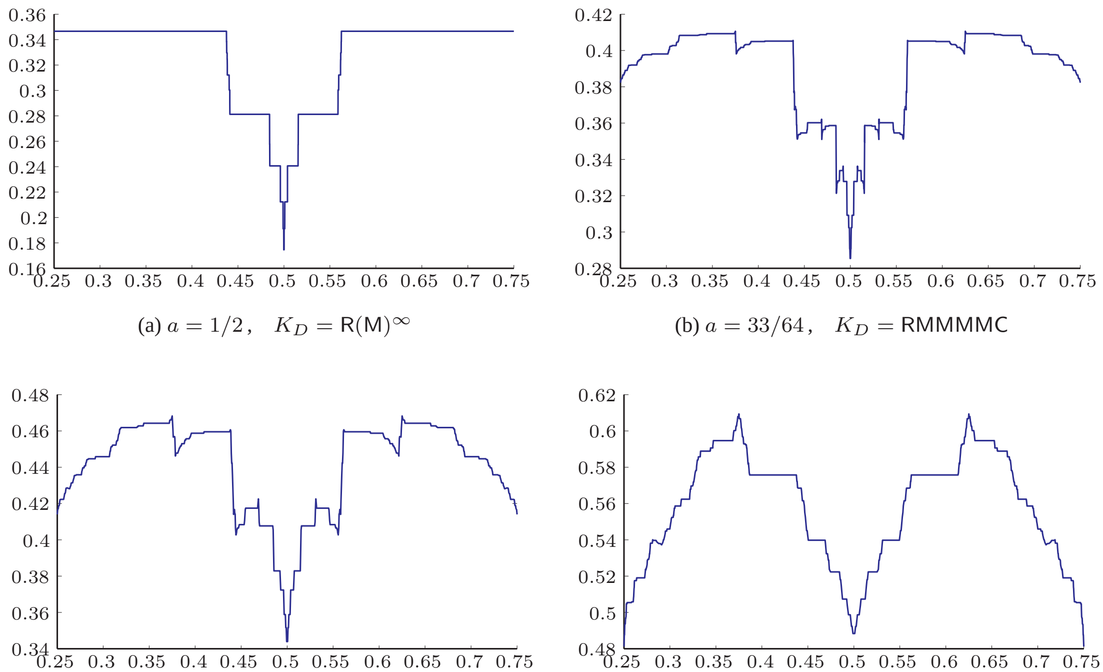


Figure 2: Piecewise linear bimodal map

To analyze the output y_i, we introduce symbolic dynamics. The unit interval $[0, 1]$, which is the invariant set of f, is divided into four regions symbolized L, M, N, and R separated by three real numbers, $D = 1/4$, θ, and $C = 3/4$. Then we define a map $A : [0, 1] \to \{\mathsf{L}, \mathsf{M}, \mathsf{N}, \mathsf{R}\}$, such that $A(x)$ is the symbol of the region in which x is located. This gives a unique symbolic itinerary $\sigma(x) = \sigma_0\sigma_1\sigma_2\ldots$ for each $x \in [0, 1]$,where $\sigma_i = A(x_i)$. We call the set of all admissible itinerary σ as the language of σ and write as L_σ or $L_\sigma(\theta; a)$ when we emphasize its dependence on the threshold θ and the parameter a. Now we introduce homomorphism $\Phi : \{\mathsf{L}, \mathsf{M}, \mathsf{N}, \mathsf{R}\} \to \{0, 1\}$,

$$\Phi(\mathsf{L}) = 0\,, \quad \Phi(\mathsf{M}) = 0\,, \quad \Phi(\mathsf{N}) = 1\,, \quad \Phi(\mathsf{R}) = 1\,. \quad (3)$$



(a) $a = 1/2\,,\quad K_D = \mathsf{R(M)}^\infty$

(b) $a = 33/64\,,\quad K_D = \mathsf{RMMMMC}$

(c) $a = 9/16\,,\quad K_D = \mathsf{RMMC}$

(d) $a = 3/4\,,\quad K_D = \mathsf{RC}$

Figure 3: Topological entropy $L_\phi(\theta; a)$ vs. θ with various a.
A horizontal axis expresses the threshold θ and a vertical axis expresses the topological entropy $L_\phi(\theta; a)$. The symbol sequence on the right of the value of a expresses the kneading sequence K_D.

This gives another unique symbolic sequence $\phi(x) = \phi_0\phi_1\phi_2\ldots$ for each $x \in [0,1]$, where $\phi_i = \Phi(\sigma_i) = \Phi(A(x_i))$. Since $y_i = \phi_i$ holds, the symbolic sequence $\phi(x)$ corresponds to the output y_i of the system. We call the set of all admissible sequence ϕ as the language of ϕ and write as L_ϕ or $L_\phi(\theta; a)$. Cleary, the language L_ϕ is not a full shift, i.e., not all the combinations of 0 and 1 are admissible. The topological entropy of a language L_ϕ is a measure how many $(0,1)$ sequences are admissible. The topological $h_\phi(\theta; a)$ entropy is defined as

$$h_\phi(\theta; a) = \limsup_{n\to\infty} \frac{\ln N_n}{n}\,, \tag{4}$$

where $N_n \le 2^n$ is the number of admissible $(0,1)$ sequences of length n in L_ϕ. Note that for $a \le 1/4$ since symbol 0(1) is surely followed by symbol 1(0), there are only two admissible sequences, $010101\ldots$ and $101010\ldots$, therefore, the topological entropy is 0. Which sequences are admissible in L_ϕ if $a > 1/4$? This problem is divided into two steps. First, we need to determine the language L_ρ with symbols $\{\mathsf{L},\mathsf{M},\mathsf{N},\mathsf{R}\}$. Second, the homomorphism Φ is applied to L_ρ to obtain L_ϕ.

Figure 3 shows the topological entropy $h_\phi(\theta; a)$ versus the threshold θ with various a. Note that these graphs have symmetry because f is symmetrical with its fixed point. It forms complicated devil's staircase-like structure, but clearly non-monotone. Moreover, the shape of the graph is very sensitive to the change in the value of the parameter a.

3. Calculation of topological entropy

As first, we need to determine the language L_σ when f is given [2, 3]. Since all four regions are either monotone increasing or monotone decreasing, the end points of the regions are important to determine the language L_ρ. We call the itinerary of $f(1/4)$, $f(\theta)$ and $f(3/4)$ as the kneading sequences of f and write as K_D, K_E and K_C respectively. Then we introduce two order relations $\prec$ and $\preceq$ between symbol sequences.

The ordering rule We have a natural order

$$\mathsf{L} \prec \mathsf{D} \prec \mathsf{M} \prec \mathsf{E} \prec \mathsf{N} \prec \mathsf{C} \prec \mathsf{R}\,, \tag{5}$$

which reflects the order of real numbers on the interval $[0,1]$. For two symbol sequence $\Sigma_1 = \Sigma\mu\cdots$ and $\Sigma_2 = \Sigma\nu\cdots$

with a common leading string Σ and the next symbols $\mu \neq \nu$, the order of Σ_1 and Σ_2 is defined as the same as the order of μ and ν if the number of M and N in Σ is even; the order of Σ_1 and Σ_2 is defined as the opposite to the order of μ and ν if the number of M and N in Σ is odd. This rule reflects that f is monotone increasing in the region L and R and monotone decreasing in the region M and N. When one of two sequences Σ_1 and Σ_2 is just the leading string of the other, the order relation $\prec$ is not defined. Then, It is convenient to define new order relation $\preceq$ such that $\Sigma_1 \preceq \Sigma_2$ holds unless $\Sigma_1 \succ \Sigma_2$. It is easy to show that $\prec$ and $\preceq$ are order relations.

Now we can determine the language L_σ.

The admissibility condition Given a bimodal map f, there is an itinerary of f that has string σ as the leading string if and only if

$$\begin{cases} & & \mathcal{L}(\sigma) & \preceq & K_D\,, \\ K_E & \preceq & \mathcal{M}(\sigma) & \preceq & K_D\,, \\ K_C & \preceq & \mathcal{N}(\sigma) & \preceq & K_E\,, \\ K_C & \preceq & \mathcal{R}(\sigma)\,, & & \end{cases} \tag{6}$$

where K_C, K_D and K_E are the kneading sequences of f, and $\mathcal{L}(\sigma)$, $\mathcal{M}(\sigma)$, $\mathcal{N}(\sigma)$, and $\mathcal{R}(\sigma)$ denote the set of the symbol sequences that follow L, M, N and R in σ respectively. The relation $\preceq$ is extended for the set such that $\mathcal{L}(\sigma) \preceq K_D$ means, for example, all sequences ρ in $\mathcal{L}(\sigma)$ satisfy $y \preceq K_D$ and so on. We can understand this rule when we consider that the region M is, for example, mapped by f to the interval $[f(\theta), f(1/4)]$, which corresponds to the condition $K_E \preceq \mathcal{M}(\sigma) \preceq K_D$.

Applying homomorphism Φ to L_σ, we determine the language L_ϕ. More precisely, we need to obtain the finite-state automaton expression of the language L_ϕ, which is required to calculate the topological entropy. For the purpose, we calculate the forbidden sequences of L_ϕ, i.e., the sequences that do not appear in L_ϕ from Eq. (6). Then, the finite-state automaton expression of the language L_ρ is constructed according to [4]. Relabeling its edges according to Eq. (3), we obtain the finite-state automaton expression of the language L_ϕ We can calculate the topological entropy of the language L_ϕ with the right-resolving expression of this finite-state automata [5].

4. Masking functions

In order to understand why the topological entropy vs. θ forms such a complicated shape, we use the masking function [6]. Note that y_i is defined as 0(or 1) if $x_i = f^i(x_0) < \theta$(or $f^i(x_0) > \theta$). When the threshold is $\theta \in [0, 1]$, the itinerary $y_i = \phi_i$ of a real number $x \in [0, 1]$ with the map f has symbol 0 as its n-th symbol if $f^{n-1}(x) < \theta$ holds. Likewise the itinerary of x has symbol 1 as its n-th symbol if $\theta > f^{n-1}(x)$ holds. Figure 4 shows the example of masking functions for symbol sequence 1101 with $a = 3/4$. For example, the dark

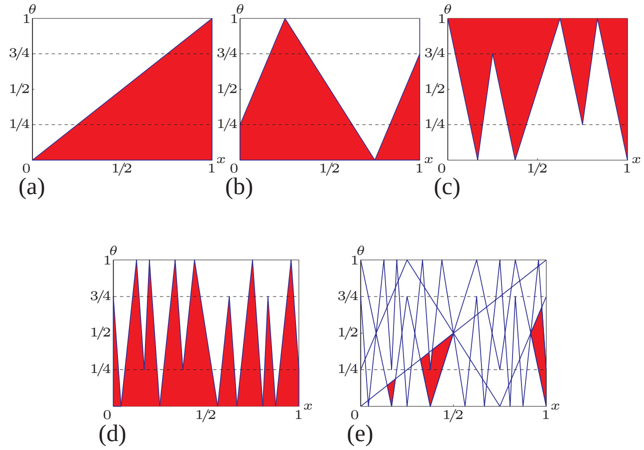


Figure 4: Masking functions for 1101.
(a) where 1 is admissible. (b) where $*1$ is admissible. (c) where $**0$ is admissible. (d) where $***1$ is admissible. (e) where the sequence 1101 is admissible. The horizontal dashed lines express the range of θ, i.e.,$1/4 \leq \theta \leq 3/4$.

region of Figure 4(a) shows the (x, θ) set with 1 as the first symbol. Likewise, the dark region of Figure 4(b) shows the (x, θ) set with 0 as the second symbol, and so on. The intersection of the dark region of Figure 4(a)$\sim$(d) is the (x, θ) set with the itinerary that has 1101 as the leading string (Figure 4(e)). Similarly, we obtain the (x, θ) set with the itinerary that begins with a given $(0, 1)$ symbolic sequence ϕ. This (x, θ) set is called as the mask for ϕ.

Given θ, a binary sequence ϕ is admissible if the mask for ϕ intersects the horizontal line of a constant θ. Therefore, the lower end and the upper end of the mask is essential for admissibility of ϕ. Then, what value of θ can be either the upper end or the lower end of a mask? Considering that the edge lines of a mask consist of the masking function $\theta = f^i(x)$ with $i = 0, 1, 2, \ldots$, there are two possibilities.

The first scenario is that the lower and the upper end θ of the mask are either the local minimum or the local maximum of $f^i(x)$ with a certain integer i. In this case, the threshold θ at either the lower or the upper end of a mask is the image of the local minimum or the local maximum value of f mapped by f several times.

Proposition 1 *Assume that a bimodal map $f(x)$ take a local minimum at $x = C$, and a local maximum at $x = D$. Then, for an integer $i > 0$, the local minimum and the local maximum value of f^i is the one of $f^j(C)$ and $f^j(D)$ with $j = 1, 2, \ldots, i$.*

Proof. For an arbitrary integer $k > 0$, assume $f^k(x)$ take

Table 1: Change of h_ϕ vs. θ with $a = 9/16$.

θ	K_E	$h_\phi(\theta; 9/16)$
$1/2$	E	0.33228546
$297/560$	MNMRE	0.40768869
$1413/2656$	MNLRE	0.41747114
$169/304$	MNLE	0.39675260
$157/280$	MNLME	0.44580831
$9/16$	MC	0.45929520
$749/1328$	MRLME	0.45929520
$701/1232$	MRLE	0.45929520
$93/152$	MRME	0.45298955
$87/140$	MRMME	0.44619725
$5/8$	D	0.46825790
$417/664$	LRMME	0.46425826
$393/616$	LRME	0.46425826
$4081/6088$	LRLE	0.46187029
$61/88$	LE	0.44580831
$55/76$	LMME	0.43584753
$26/35$	LMMME	0.42228451
$3/4$	LMMD	0.41401273

either local minimums or local maximums ζ_k at $x = \xi_k$, i.e. $\zeta_k = f^k(\xi_k)$ holds. Thus,

$$\begin{aligned} 0 &= \left(f^i(\xi_i)\right)' \\ &= f'(f^{i-1}(\xi_i))\, f'(f^{i-2}(\xi_i)) \cdots f'(f(\xi_i))\, f'(\xi_i)\,. \end{aligned} \tag{7}$$

This equation is satisfied if $f^j(\xi_i) = \xi_1$ holds with a certain $0 \le j < i$. Thus, $\zeta_i = f^i(\xi_i) = f^{i-j}(\xi_1)$. □

For symbolic dynamics, the kneading sequences K_D and K_C, which are the itinerary of $f(D)$ and that of $f(C)$, reflect this scenario.

The second possible scenario is that the lower and the upper end of a mask are the intersection points of two curves $\theta = f^i(x)$ and $\theta = f^j(x)$ with integers $i \ne j$.

Proposition 2 *Given a bimodal map f and integers $i > j$, assume that θ is the value of the intersection point of f^i and f^j, i.e., $\theta = f^i(x) = f^j(x)$. Then θ is a fixed point of f^{i-j}, which satisfies $\theta = f^{i-j}(\theta)$.*

Proof. For an arbitrary integer $k > 0$, x_k denotes the fixed point of x_k, i.e., $x_k = f^k(x_k)$. Since

$$\begin{aligned} \theta &= f^i(x) &= f^j(x) \\ &= f^{i-j}(f^j(x)) &= f^j(x)\,, \end{aligned} \tag{8}$$

$f^j(x)$ is the fixed point of f^{i-j}. Therefore $\theta = f^j(x) = x_{i-j}$.

□

In this case, it is enough to consider the mask that shares one edge with the line $\theta = x$. Assume a mask dose not share any edge with $\theta = x$. The corner points of the mask are the intersection point of $\theta = f^i(x)$ and $\theta = f^j(x)$ with some $i > j > 0$. Due to Proposition 2, such θ satisfies $\theta = f^{i-j}(x)$, which is the intersection point of $\theta = x$ and $\theta = f^{i-j}(x)$. Therefore there exists another mask that shares one edge with line $\theta = x$ such that the set of the values θs of its corner points are the same of the previous mask.

In this sense, the fixed points and periodic points of f may be critical values for change in topological entropy vs. threshold. If the threshold θ crosses the value of fixed points and periodic points of f, the language $L_\phi(\theta; a)$ may change. Thus, the devil's staircase of the topological entropy has its steps at the fixed points and the periodic points of f.

Table 1 shows the topological entropy h_ϕ vs. threshold θ with the parameter value $a = 9/16$ (cf. Figure 3(c)). We choose θ as the periodic point of f such that their length are five or less, and the images of $f(C)$ and $f(D)$. Note that from the symmetry of f, only the half range of θ $1/2 \le \theta \le 3/4$ is shown. The topological entropy h_ϕ may change, in other word, the language L_ϕ may change, at these points.

Conclusion

We have applied symbolic dynamics to the chaotic neuron model to analyze the effect of the threshold and found the topological entropy vs. threshold level forms a devil's staircase-like structure, but non-monotone. This devil's staircase has its steps at the fixed points and the periodic points of the map.

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