International Symposium on Nonlinear Theory and its Applications
Xi'an, PRC, October 7–11, 2002

A Fast Iterative Method with Algebraic Reconstruction Technique for Computerized Tomography

Tetsuya Yoshinaga

Department of Radiologic Science and Engineering
School of Health Sciences, The University of Tokushima
3-18-15 Kuramoto, Tokushima, 770-8509 Japan
Phone: +81-88-633-9050, Fax: +81-88-633-9015
E-mail: yosinaga@medsci.tokushima-u.ac.jp

Abstract

Considering a system of simultaneous iterative reconstruction technique (SIRT) for X-ray computerized tomography (CT) as a discrete dynamical system, the reconstruction process can be reduced to a procedure of finding a fixed point of the dynamical system. We examine a numerical method for solving fixed points of dynamical systems derived from the algebraic reconstruction technique (ART) and the expectation maximization (EM) formulation, giving rise to the very fast convergence for CT reconstruction. Because the proposed method is based on the SIRT, it has an advantage for reducing metal artifact against the filtered backprojection procedure.

1. Introduction

The simultaneous iterative reconstruction technique (SIRT)[1, 2, 3] is known as a method to reconstruct images of X-ray computerized tomography (CT). Iterative deblurring methods using the algebraic reconstruction technique (ART)[1] and the expectation maximization (EM)[4] formulation have advantages for reducing artifacts against the filtered backprojection procedure, which is commonly used for CT reconstruction in practice. Because of the high-quality reconstructions, there has been a lot of research[5, 6, 7] improving the SIRT procedures. However, each type of SIRT requires a large number of iterations to obtain the final reconstruction image.

On the other hand, the iterative formulae of SIRT can be considered as a dynamical system of reconstruction grid elements in the N-dimensional state space, where N is the number of pixels. Then a stable fixed point of the system corresponds to the image to be reconstructed. Moreover the dynamics in the neighborhood of the fixed point govern the convergence speed of the SIRT procedure. In this paper, we survey properties of the system from the viewpoint of dynamical system theory, and examine a numerical method for solving a fixed point of the system, using a shooting method such as Newton's method. The proposed method gives rise to the very fast convergence for the reconstruction[8]. The efficiency is illustrated using numerical experiments with synthetic additive noise projection data from a phantom including a metal part.

2. Dynamical System and Its Fixed Point

Consider an N-dimensional discrete dynamical system

$$u^{(k+1)} = T(u^{(k)}), \quad k = 1, 2, \ldots \tag{1}$$

or a mapping defined by

$$T \ : \ R^N \to R^N \ ; \ u \mapsto T(u) \tag{2}$$

where $u^{(k)}$ and u are state vectors in R^N. The point u^* satisfying

$$F(u^*) := u^* - T(u^*) = 0 \tag{3}$$

becomes a fixed point of T. Let u^* be a fixed point, then the characteristic equation of the fixed point u^* is defined by

$$\det(\mu I - DT(u^*)) = 0 \tag{4}$$

where I is the $N \times N$ identity matrix, and DT denotes the derivative or the Jacobian of T with respect to the initial state. u^* is hyperbolic, if all absolute values of the eigenvalues of T are different from unity. The stability of a hyperbolic fixed point is determined by $\dim E$, the intersection of R^N and the direct sum of the generalized eigenspaces of $DT(u^*)$ corresponding to the eigenvalues μ such that $|\mu| > 1$. The fixed point u^* is stable if and only if the dimension of the unstable subspace $\dim E$ is zero, or every eigenvalue satisfies $|\mu| < 1$.

Let us consider a method for finding the fixed point $u = u^*$. If a fixed point u^* is hyperbolic, then the method is accomplished by solving Eq.(3). Under the assumption, the Jacobian matrix of F becomes nonsingular. Hence we can apply Newton's method for solving Eq.(3).

Assuming u^* be an isolated solution of Eq.(3), we have an iterative algorithm as follows:

1. Choose an initial guess $u^{(0)}$, i.e., an approximate solution $u^{(0)}$ of u^*.

2. Compute $u^{(k+1)}$ from $u^{(k)}$ by the following relation:

$$u^{(k+1)} = u^{(k)} + \lambda h \tag{5}$$

$$DF(u^{(k)})h + F(u^{(k)}) = 0 \tag{6}$$

where λ is a deceleration factor in $(0, 1]$ and is normally set to 1. The exceptional variation of λ will be discussed later.

3. Repeat until

$$\|u^{(k+1)} - u^{(k)}\| < \varepsilon_u \quad \text{or} \quad \|F(u^{(k+1)})\| < \varepsilon_F \tag{7}$$

is satisfied, where ε_u and ε_F are prescribed small positive constants.

3. Iterative Reconstruction

We should summarize the iterative image reconstruction algorithms. The value of the acquired projection is given by

$$p(u^*, \theta, t) = \sum_{j=1}^{N} u_j^* \delta(x_j \cos\theta + y_j \sin\theta - t) \tag{8}$$

where N is the number of pixels, $u^* = (u_1^*, u_2^*, \ldots, u_N^*)'$ is a set of the reconstruction grid elements, each of which, u_j^*, is on the coordinate (x_j, y_j) of the jth pixel for $j = 1, 2, \ldots, N$, θ denotes the projection orientation, and t indicates the detection position. Strictly speaking, because image values and projection data are located on two- and one-dimensional grids, respectively, δ must be the delta function of an appropriate interpolated position. Our task is to estimate the unknown u^*.

We now define the projection through the point (x_i, y_i) for $i = 1, 2, \ldots, N$, as

$$p_i(u^*, \theta) = \sum_{j=1}^{N} u_j^* \delta((x_j - x_i)\cos\theta + (y_j - y_i)\sin\theta) \tag{9}$$

and a synthesized projection through the same point, based on a set of current guesses $u = (u_1, u_2, \ldots, u_N)'$ as

$$p_i(u, \theta) = \sum_{j=1}^{N} u_j \delta((x_j - x_i)\cos\theta + (y_j - y_i)\sin\theta) \tag{10}$$

Then the ART- and EM-type SIRT algorithms can be written in the mapping form of Eq.(2) with the following elements T_i's for $i = 1, 2, \ldots, N$.

- In the case of ART:

$$T_i \ : \ R^N \to R \ ; \ u \mapsto T_i(u) = u_i + f_i(u) \tag{11}$$

where

$$f_i(u) = \frac{1}{n_i^2} \int [p_i(u^*, \theta) - p_i(u, \theta)] d\theta \tag{12}$$

- In the case of EM:

$$T_i \ : \ R^N \to R \ ; \ u \mapsto T_i(u) = u_i g_i(u) \tag{13}$$

where

$$g_i(u) = \frac{1}{n_i} \int \frac{p_i(u^*, \theta)}{p_i(u, \theta)} d\theta \tag{14}$$

Here n_i in Eqs.(12) and (14) is

$$n_i = \int \sum_{j=1}^{N} \delta((x_j - x_i)\cos\theta + (y_j - y_i)\sin\theta) d\theta \tag{15}$$

Note that the dynamical systems $T(u)$ for ART and EM are linear and nonlinear with respect to the state variable u, respectively.

To implement the method for finding the fixed point of T, the (i, j) element of the matrix DT is required for ART:

$$\frac{\partial T_i(u)}{\partial u_j} = \begin{cases} \dfrac{\partial f_i(u)}{\partial u_j} & (i \neq j) \\ 1 + \dfrac{\partial f_i(u)}{\partial u_j} & (i = j) \end{cases} \tag{16}$$

with

$$\begin{aligned} \frac{\partial f_i(u)}{\partial u_j} &= -\frac{1}{n_i^2} \int \frac{\partial p_i(u, \theta)}{\partial u_j} d\theta \\ &= -\frac{1}{n_i^2} \int \delta((x_j - x_i)\cos\theta + (y_j - y_i)\sin\theta) d\theta \end{aligned}$$

and for EM:

$$\frac{\partial T_i(u)}{\partial u_j} = \begin{cases} u_i \dfrac{\partial g_i(u)}{\partial u_j} & (i \neq j) \\ g_i(u) + u_i \dfrac{\partial g_i(u)}{\partial u_j} & (i = j) \end{cases} \tag{17}$$

with

$$\begin{aligned} \frac{\partial g_i(u)}{\partial u_j} &= -\frac{1}{n_i} \int \frac{p_i(u^*, \theta)}{p_i^2(u, \theta)} \cdot \frac{\partial p_i(u, \theta)}{\partial u_j} d\theta \\ &= -\frac{1}{n_i} \int \frac{p_i(u^*, \theta)}{p_i^2(u, \theta)} \delta((x_j - x_i)\cos\theta + (y_j - y_i)\sin\theta) d\theta \end{aligned}$$

We note that the procedure for finding a fixed point can be applied to the case of the presence of a metal part, in the same manner as in Ref.[7]. Namely, the metal artifact reduction can be done by replacing the whole interval of integration with respect to θ in $[0, \pi]$ with a restricted range Θ_i such that

$$\Theta_i = \{\theta \in [0, \pi] \mid p_i(u^*, \theta) \neq 0\} \tag{18}$$

This condition is derived from the fact that any cross section passing through the metal part is not measurable or the projection is vanished.

4. Analytical Results

We show properties of the dynamical systems derived from the SIRT algorithms, and then illustrate a convergence to their fixed points.

4.1. Theoretical Analysis

First, we show the reason why the convergence speed of the EM algorithm is faster than that of the ART algorithm[7]. As described in Section 2, $DT(u^*)$ plays an important role on the stability of the fixed point u^* of T. For the case of ART, the (i, j) element of the matrix $DT(u^*)$ becomes

$$\frac{\partial T_i}{\partial u_j}(u^*) = \begin{cases} -\dfrac{1}{n_i^2}\displaystyle\int \delta((x_j - x_i)\cos\theta \\ \qquad\qquad +(y_j - y_i)\sin\theta)d\theta & (i \neq j) \\ 1 - \dfrac{1}{n_i} & (i = j) \end{cases} \tag{19}$$

We note that since $n_i \gg 1$ and

$$-\frac{1}{n_i} < \frac{\partial T_i}{\partial u_j}(u^*) < -\frac{1}{n_i^2} \tag{20}$$

for $i \neq j$, every eigenvalue of $DT(u^*)$ is less than and around 1. This shows that the point u^* is stable and the convergence speed is relatively slow.

On the other hand, the fixed point u^* of the mapping T for EM is globally stable[5] and the eigenvalues of $DT(u^*)$ include 0. Indeed, we can prove that an eigenvector with respect to the zero eigenvalue is u^*, i.e.

$$DT(u^*)u^* = u^* + \left(u_i^* \sum_{j=1}^{N} \frac{\partial g_i(u^*)}{\partial u_j} u_j^* \right)_{i=1,2,\ldots,N} = 0 \tag{21}$$

because $g_i(u^*) = 1$ and

$$\begin{aligned} \sum_{j=1}^{N} \frac{\partial g_i(u^*)}{\partial u_j} u_j^* &= -\frac{1}{n_i} \sum_{j=1}^{N} \int \frac{p_i(u^*, \theta)}{p_i^2(u^*, \theta)} \cdot u_j^* \frac{\partial p_i(u^*, \theta)}{\partial u_j} d\theta \\ &= -\frac{1}{n_i} \int \frac{1}{p_i(u^*, \theta)} \sum_{j=1}^{N} u_j^* \delta((x_j - x_i)\cos\theta \\ &\qquad\qquad +(y_j - y_i)\sin\theta)d\theta \\ &= -1 \end{aligned}$$

for $i = 1, 2, \ldots, N$. Therefore it can be said that the EM algorithm generally converges to the fixed point faster than the ART algorithm.

Next, we consider the procedure for getting fixed point using Newton's method. The system of ART in Eq.(11) is linear with respect to the state variable u, therefore, in the case of nonexistence of metal, the convergence to the fixed point u^* is accomplished by one step of Newton's method starting from any initial state. While considering the application to the system of EM, a good first guess $u^{(0)} > 0$ or an approximate state of u^* should be obtained, e.g., by processing several steps of the direct EM iterations; and for keeping nonnegative value of $u^{(k+1)}$, the deceleration factor λ in Eq.(5) should be considered in every kth step. For example, it can be done by replacing the value of λ by its half successively while there exists a negative $u_i^{(k+1)}$ for $i = 1, 2, \ldots, N$.

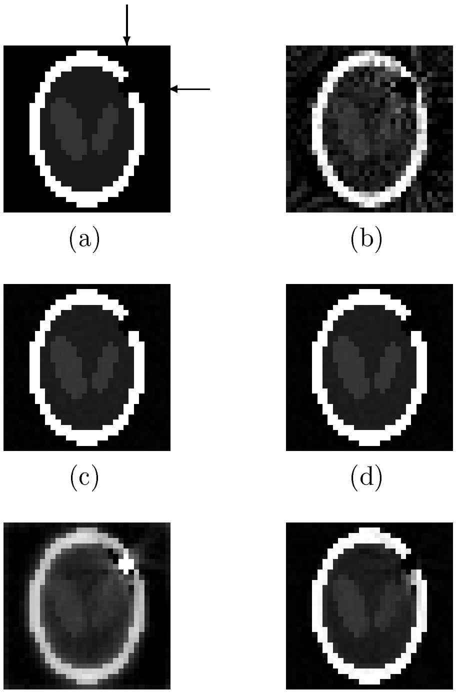
4.2. Numerical Experiments

To illustrate the efficiency of the procedure for finding the fixed point of the dynamical systems in Eqs.(11) and (13), numerical experiments were examined.

Reconstruction was done using 100 detectors per projection and 180 degrees scanning. The initial guess for the ART and EM iterations was set to a constant image (1.2, in our simulation).

We demonstrate the effectiveness of metal artifact reduction with noisy projection data. Figure 1 shows simulation results for an image of 32×32 pixels, where (a) is the phantom image with metal in a small circle whose center is denoted by the intersection point of two arrows; (b) is the image reconstructed through filtered backprojection with noise-free projection data; (c) and (d) are images reconstructed using Newton's method applied to the fixed point equation derived from the ART and EM methods, respectively, with noisy projection data; and (e) and (f) are counterparts of (c) and (d) using the ART and EM methods, respectively. The noise added to the projection data came from a random uniform distribution in $[0, 0.02]$, where the value 0.02 is 2% of the maximum density of the image. In (c), the exact solution was solved by one step in the meaning of $\varepsilon_F = 10^{-4}$ in Eq.(7) with the I-divergence[7, 5] d, as a distance norm:

$$d = \int p_i(u^*, \theta) \log \frac{p_i(u^*, \theta)}{p_i(u, \theta)} d\theta - \int [p_i(u^*, \theta) - p_i(u, \theta)] d\theta \tag{22}$$



(a) (b) (c) (d) (e) (f)

Figure 1: Simulation for metal artifact reduction with noisy projection data: (a) phantom image, (b) image reconstructed through filtered backprojection; (c) and (d) images reconstructed using Newton's method applied to the fixed point equations derived from ART (with 1 step) and EM (with three steps preceded by 30 iterations of the direct EM method), respectively; (e) and (f) counterparts of (c) and (d) using the methods of ART (with 60 iterations) and EM (with 60 iterations), respectively.

The image (d), whose I-divergence is also less than 10^{-4}, was obtained by three steps of Newton's method preceded by 30 iterations of the EM process. If there is no free-run process for approaching the fixed point, the value of the state variable generally diverges. The I-divergences of the images (e) and (f) against the original image (a) are 97.0 and 1.57, respectively. We see that (c) and (d) are better than (e) and (f). The minimum and the maximum eigenvalues of $DT(u^*)$ for ART and EM in some example are listed in Table 1. We see that the convergence speed is related to the eigenvalues.

5. Concluding Remarks

We have investigated the dynamical systems that come from the ART and EM iterative reconstruction techniques. The main results obtained in this paper are summarized as follows:

Table 1: Eigenvalues of $DT(u^*)$ for the ART- and EM-type SIRT

SIRT	eigenvalues	
	minimum	maximum
ART	0.965	0.994
EM	0	0.948

1. The dynamics of the reconstruction techniques have been clarified by studying properties of eigenvalues of $DT(u^*)$. The convergence speed is essentially governed by the properties such that all eigenvalues for ART are less than and around 1, and eigenvalues for EM include 0.

2. We have shown that the method for solving the fixed point equation by the use of Newton's method is efficient for the image reconstruction. Especially, we can get the exact image by one step from any initial value using the method with ART.

3. We have illustrated that the procedure for finding the fixed point with ART can be applied to the case of the presence of a metal part with noisy projection data and that it produces a good reduction of metal artifacts.

References

[1] R. Gordon, R. Bender and G.T. Herman, Algebraic reconstruction technique (ART) for three-dimensional electron microscopy and X-ray photography, *J. Theoret. Biol.*, **29**, (1970), 471–482.

[2] A.C. Kak and M. Slaney, *Principles of Computerized Tomographic Imaging.* Piscataway, NJ: IEEE Press, 1987.

[3] H. Stark, *Image Recovery: Theory and Application.* FL: Academic, 1987.

[4] L.A. Shepp and Y. Vardi, Maximum likelihood reconstruction in positron emission tomography, *IEEE Trans. Med. Imag.*, **1** (1982), 113–122.

[5] D.L. Snyder, T.J. Schulz and J.A. O'Sullivan, Deblurring subject to nonnegativity constraints, *IEEE Trans. Signal Processing*, **40-5** (1992), 1143–1150.

[6] G.T. Herman and L.B. Meyer, Algebraic reconstruction techniques can be made computationally efficient, *IEEE Trans. Med. Imag.*, **12-3** (1993), 600–609.

[7] G. Wang, D.L. Snyder, J.A. O'Sullivan and M.W. Vannier, Iterative deblurring for CT metal artifact reduction, *IEEE Trans. Med. Imag.*, **15-5** (1996), 657–664.

[8] T. Yoshinaga, A fast convergence method with simultaneous iterative reconstruction technique for computerized tomography, *Int. J. Imaging Systems and Technology*, **10-6** (2000), 432–436.