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The Bifurcation Set of Swollen Shape Type Bifurcation

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Abstract

In this paper, we analyze a kind of torus doubling, which we call swollen shape type bifurcation, in detail. It is observed in a coupled system of a logistic map and a sin circle map. In this system, though it is difficult to trace theoretically the existence of swollen shape type bifurcation, it is possible to trace theoretically co-dimension 2 bifurcation point. We traced co-dimension 2 bifurcation point. If this bifurcation point occurs successively, it can be thought that it is possible to become powerful grounds that swollen shape type bifurcation occurs successively. As a result, though we find out a lot of co-dimension 2 bifurcation point with double precision, parameters of co-dimension 2 bifurcation point oscillate. It is clarified that an universal constant corresponding to the Feigenbaum's universal constant is not obtained.

Keywords— torus, torus doubling, co-dimention 2 bifurcation.

1. Introduction

TORUS doubling is known as one of the most interesting transitions from torus to chaos. According to some reports[1, 2], torus doubling occurred finite times. It was considered that a torus oscillated, and that chaos was generated after finite torus doublings. Kaneko proposed a coupled system of a logistic map and a linear circle map, and asserted that the basic mechanism of torus doubling can be explained by this system[1, 2]. He showed by detailed numerical analysis that torus doubling occurred only finite times in this system. He explained that the fixed-point function of Feigenbaum is unstable against perturbation of torus[2].

We found a certain kind of torus doubling, which we call "swollen shape type bifurcation," occurs successively in the neighborhood of regions where a torus is resonant[3]. Torus observed after swollen shape type bifurcation has very delicate structure. Nevertheless, this phenomenon is observed in an actual laboratory experiment. Though Kaneko proposed a coupled system of a logistic map and a circle map, liner circle map is employed for computer simulation[1, 2]. Since the swollen shape type bifurcation occurs in the neighborhood of the regions where a torus is resonant, it is considered that this phenomenon has a deep relation to Arnold's tongues. We considered that the coupled system which Kaneko proposed is inappropriate model which clarifies the whole mechanism of torus doubling. Therefore, we proposed a coupled system of a logistic map and a sin circle map. By numerical analysis, the swollen shape type bifurcation is also observed in this coupled system. In addition, we made bifurcation diagram roughly, we suggested that accumulating point of swollen shape type bifurcation is co-dimension 2 bifurcation point[3].

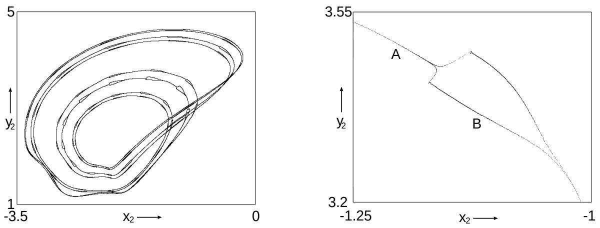
In this paper, we analyze swollen shape type bifurcation, which is observed in a coupled system of a logistic map and a sin circle map, in detail. In this system, though it is difficult to trace theoretically the existence of swollen shape type bifurcation, it is possible to trace theoretically co-dimension 2 bifurcation point. An efficient algorism is proposed by Kawakami[4, 5]. Because the bifurcation equation of this co-dimension 2 bifurcation is shown by transcendental equation, it can be solved by the arbitrary accuracy. We traced the co-dimension 2 bifurcation point. If this bifurcation point occurs successively, it can be thought that it is possible to become powerful grounds that swollen shape type bifurcation occurs successively. In addition, if an universal constant corresponding to the Feigenbaum's universal constant exists, it is possible to become grounds that torus doubling occurs infinite times.

We traced co-dimension 2 bifurcation point with double precision. As a result, though we find out a lot of co-dimension 2 bifurcation point, parameters of co-dimension 2 bifurcation point oscillate. It is clarified that an universal constant corresponding to the Feigenbaum's universal constant is not obtained.

2. Summary of [3]

We showed that a kind of torus doubling, which we call "swollen shape type bifurcation," occurs successively in the neighborhood of regions where a torus is resonant in [3].

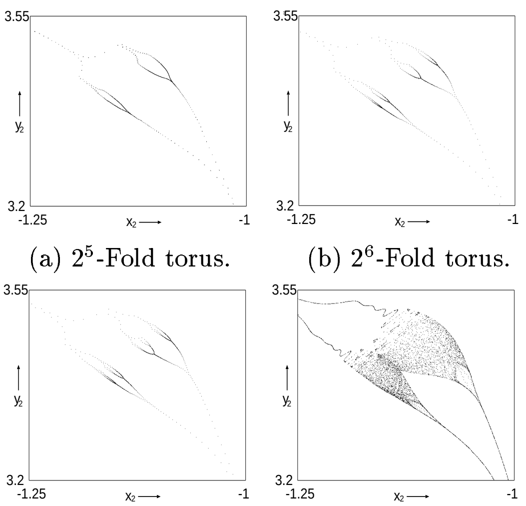
Fig. 1 shows attractors on Poincaré section of a cer-



(a) A global picture. (b) An enlargement of (a).

Figure 1: Attractors on Poincaré section.

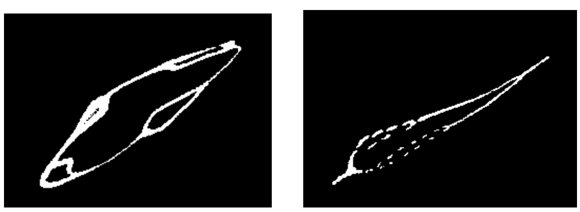
tain 4-dimensional self-excited oscillator by numerical simulation. This figure shows attractor of first swollen shape type bifurcation. The curve which forms torus approaches each other considerably in the neighborhood of the point marked with "A" in Fig. 1(b), and the attractor has a swelling shape in the neighborhood of the point marked with "B." This phenomenon is observed in the neighborhood of regions where a torus is resonant. Fig. 2 shows the successive swollen shape type



(a) 2^5-Fold torus. (b) 2^6-Fold torus.

(c) 2^7-Fold torus. (d) Chaos.

Figure 2: Attractors on Poincaré section.



(a) 2-Fold torus. (b) 4-Fold torus.

Figure 3: Attractors on Poincaré section.

bifurcation. Swollen shape type bifurcation is also observed in an actual laboratory experiment, though it has a very delicate structure. Fig. 3(a) shows 2-Fold torus which is observed after the first swollen shape type bifurcation. The successive bifurcation phenomenon is also observed in the laboratory experiment. Fig. 3(b) shows the 4-Fold torus which is observed after second swollen shape type bifurcation.

In addition, we made the rough bifurcation diagram shown in Fig. 4, we suggested that the accumulation point of swollen shape type bifurcation will be co-dimension 2 bifurcation point. The region labeled "resonant" shows the region where frequency locking occurs, the region labeled "chaos" shows the region where chaos occurs, "S_1" is a region where a 2^4-Fold torus occurs after the first swollen shape type bifurcation, "S_2" is a region where a 2^5-Fold torus occurs, "S_3" is a region where a 2^6-Fold torus occurs, and "S_4" is a region where a 2^7-Fold torus occurs.

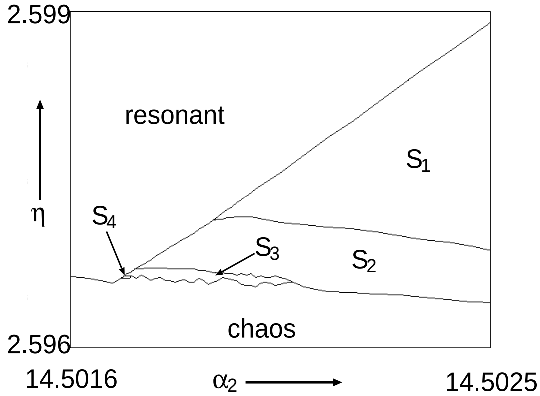
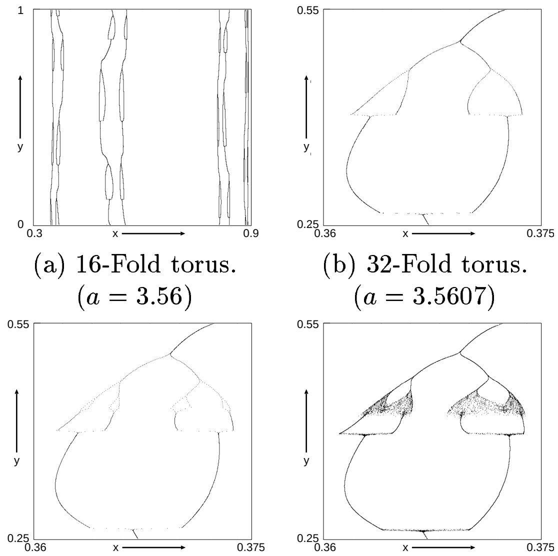


Figure 4: Bifurcation diagram.

As mentioned above, swollen shape type bifurcation occurs in the neighborhood of regions where a torus is resonant. It is considered that this phenomenon has a deep relation to Arnold's tongues. Therefore, a following coupled system of a logistic map and a sin circle map is proposed as the model which can clarify the mechanism of torus doubling:

$$\left\{\begin{array}{rcl} x_{n+1} & = & ax_n(1-x_n)+\varepsilon_1 f(x_n,\ y_n) \\ y_{n+1} & = & y_n + b\sin 2\pi y_n + c + \varepsilon_2 g(x_n,\ y_n) \\ & & \qquad\qquad\qquad\qquad (\mathrm{mod}\ 1) \end{array}\right.$$
$$f(x_n,\ y_n) = \sin 2\pi y_n, \qquad g(x_n,\ y_n) = x_n, \tag{1}$$

where a, b and c are constants, ε_1 and ε_2 are the uniting constants. Fig. 5 shows swollen shape type bifurcation on x_n–y_n plane.



(a) 16-Fold torus. ($a = 3.56$)

(b) 32-Fold torus. ($a = 3.5607$)

(c) 64-Fold torus. ($a = 3.560792$)

(d) Chaos. ($a = 3.56085$)

Figure 5: The swollen shape type bifurcation. ($b = 0.1, c = 0.419098576$, $\varepsilon_1 = 0.0007$, $\varepsilon_2 = 0.0007$.)

3. Analysis method

The method shown in [4] is used for the analysis. It is summarized as follows. Equation (1) is equivalent to the following equation:

$$\boldsymbol{x}_{n+1} = \mathrm{T}(\boldsymbol{x}_n). \tag{2}$$

When $\boldsymbol{x}_0$ is periodic point with period k, equation (1) satisfies the following equation:

$$\mathrm{T}^k(\boldsymbol{x}_0) - \boldsymbol{x}_0 = 0, \tag{3}$$

where $\mathrm{T}^k(\boldsymbol{x}_0)$ denotes k times composition map of T. The characteristic of a periodic point $\boldsymbol{x}_0$ is classified by the root μ of following characteristic equation:

$$\left| \frac{d}{d\boldsymbol{x}_0} \mathrm{T}^k(\boldsymbol{x}_0) - \mu \boldsymbol{I} \right| = 0. \tag{4}$$

Let the roots of the characteristic equation be denoted by μ_1 and μ_2 ($|\mu_1| \geq |\mu_2|$). If $|\mu_1| < 1$, a periodic point is called stable. If $|\mu_1| > 1 > |\mu_2|$, a periodic point is called saddle. If $|\mu_2| > 1$, a periodic point is called completely unstable. At $|\mu| = 1$, a bifurcation occurs. In the analyses of equation (1), following bifurcations are observed:

- Saddle node bifurcation: This bifurcation occurs when $\mu = 1$. A saddle and a stable periodic point, or a saddle and a completely unstable periodic point are merged and these periodic points disappear, when we change the parameter to pass the bifurcation set. In the following discussion, this bifurcation is denoted by 'G.'
- Period doubling bifurcation: This bifurcation occurs when $\mu = -1$. A stable n-periodic point becomes a saddle. On both sides of this saddle, a stable periodic point with period $2n$ is generated. Period doubling bifurcation occurs successively. Chaos occurs after the accumulation of period doubling bifurcation. In the following discussion, this bifurcation is denoted by 'I.'

4. Bifurcation phenomena

We fix parameters as $b = 0.1, \varepsilon_1 = 0.0007$, $\varepsilon_2 = 0.0007$, and derive a bifurcation diagram on the $c-a$ plane. Fig. 6 shows a global picture of bifurcation diagram. We can get saddle node bifurcation curve marked 'G^k' by solving simultaneous equations shown in equation (5). We can get period doubling bifurcation curve marked 'I^k' by solving simultaneous equations shown in equation (6). 'k' is the order of period. And, we can get co-dimension 2 bifurcation point which is marked with '●' by solving simultaneous equations shown in equation (7). Fig. 7 show figure where the co-dimension 2 bifurcation point is generated.

$$\left\{ \begin{array}{lcl} \mathrm{T}^k(\boldsymbol{x}_0) - \boldsymbol{x}_0 & = & 0 \\ \left| \dfrac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}_0} \mathrm{T}^k(\boldsymbol{x}_0) - \boldsymbol{I} \right| & = & 0 \end{array} \right. \tag{5}$$

$$\left\{ \begin{array}{lcl} \mathrm{T}^k(\boldsymbol{x}_0) - \boldsymbol{x}_0 & = & 0 \\ \left| \dfrac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}_0} \mathrm{T}^k(\boldsymbol{x}_0) + \boldsymbol{I} \right| & = & 0 \end{array} \right. \tag{6}$$

$$\left\{ \begin{array}{lcl} \mathrm{T}^k(\boldsymbol{x}_0) - \boldsymbol{x}_0 & = & 0 \\ \left| \dfrac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}_0} \mathrm{T}^k(\boldsymbol{x}_0) - \boldsymbol{I} \right| & = & 0 \\ \left| \dfrac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}_0} \mathrm{T}^k(\boldsymbol{x}_0) + \boldsymbol{I} \right| & = & 0 \end{array} \right. \tag{7}$$

Parameter values of the co-dimension 2 bifurcation point are as follows;

$$\begin{aligned} a_0 &= 3.54505176573635\cdots, \\ a_1 &= 3.56279230786907\cdots, \\ a_2 &= 3.56354136074252\cdots, \\ a_3 &= 3.56369141601959\cdots, \\ a_4 &= 3.56372348255832\cdots, \\ a_5 &= 3.56373015697973\cdots. \end{aligned}$$

We define constant δ_n shown in equation (8).

$$\delta_n = \frac{a_{n+1} - a_n}{a_{n+2} - a_{n+1}} \tag{8}$$

a_n is parameter value of co-dimension 2 bifurcation point. If δ_n converges, it can be thought that it is possible to become powerful grounds that torus doubling

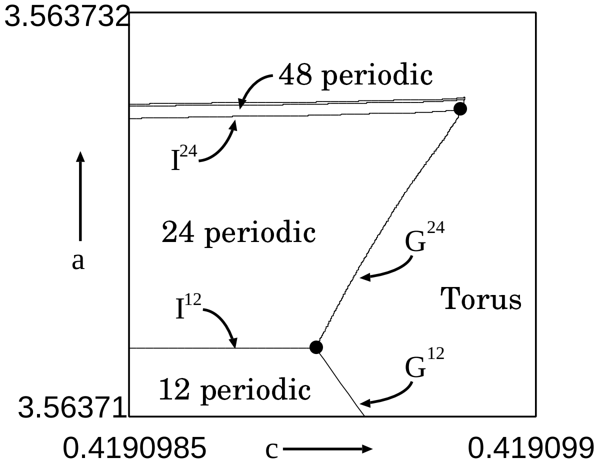


Figure 6: A global picture of the bifurcation diagram.

occurs successively. But, as shown in Table 1, parameter values of co-dimension 2 bifurcation point oscillate. We cannot get universal constant.

Table 1: δ_n

n	δ_n
1	23.683965
2	4.9918462
3	4.6794971
4	4.8043922

5. Conclusion

In this paper, we trace co-dimension 2 bifurcation point with computer with double precision. As a result, though we find out a lot of generation, parameter of co-dimension 2 bifurcation point oscillates. It is clarified that an universal constant corresponding to the Feigenbaum's universal constant is not obtained.

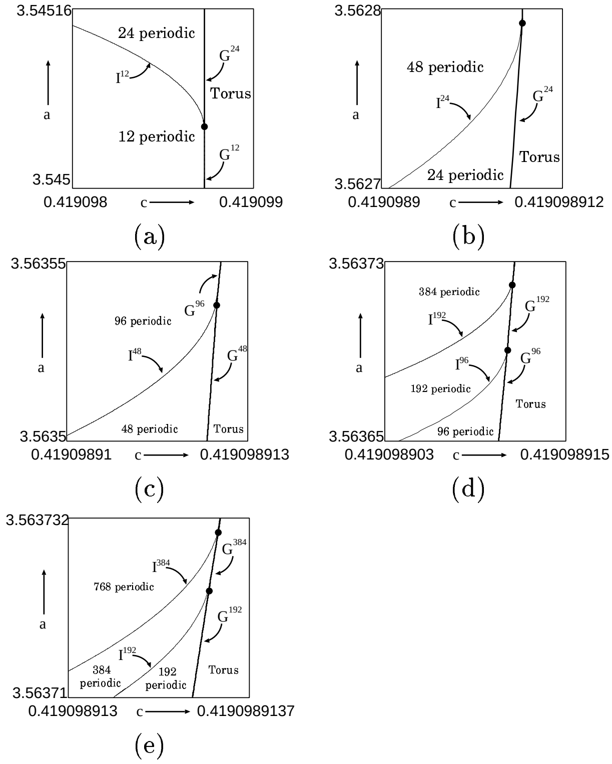


Figure 7: An enlargement of neighborhood of co-dimention 2 bifurcation point.

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