# Scattering Matrix Method for Transient Analysis of Hybrid System Composed of Liner/Nonlinear Lumped Elements and Frequency Dependent Lossy Transmission Lines with Distributed Sources

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**Abstract** An efficient method is presented to calculate the transients in the hybrid system composed of multi-conductor transmission lines with frequency dependent losses and linear/nonlinear lumped elements. Each transmission line may be either uniform or nonuniform in its configuration of conductors and immersed in incident fields. A straightforward matrix formulation by the scattering matrix method is derived by Laplace transformation and inverted into the time domain by the numerical method. Lumped portions are formulated by nonlinear equations. The overall equations governing the whole system are discretized and solved numerically in the time domain.

## 1 Introduction

VLSI is the complicated hybrid system composed of the large number of linear/nonlinear lumped elements and distributed interconnects. The increase in clock frequencies within VLSI produces high frequency components in signal transients. At these high frequencies the interconnects are electrically long relative to the wavelength of the signal and accurate modeling of interconnects by distributed transmission lines is no longer avoidable. For the interconnects with arbitrary geometries, transmission lines are either uniform or nonuniform. In some cases, the interconnects may be immersed in incident fields. So VLSI is the hybrid system composed of lumped elements and distributed transmission lines.

In high frequency applications the loss of interconnect is determined by the skin effect of the conductor. In the complex frequency domain, this loss can be handled as the square root dependency on the Laplace operator $s$.

The objective of this paper is to present a generalized simulation method for complicated hybrid system such as VLSI. For the portion composed of transmission lines, a straightforward matrix formulation by the scattering matrix method is derived by Laplace transformation and inverted into the time domain by the numerical method. Lumped portions are formulated by nonlinear equations. The overall equations governing the whole system are discretized and solved numerically in the time domain. We show the efficient method which provides precise results in the practical applications.

## 2 Computational model of transmission line with distributed source

### 2.1 Single uniform line in complex frequency domain

Consider a $N_m+1$ conductor uniform transmission line labeled $\sharp m$ as shown in Fig.1 where the conductor 0 is the reference line. The transmission line equations for $N_m \times 1$ voltage and current vectors

$$\boldsymbol{V}_m(z,s) = [V_{m,1}(z,s)\ \ V_{m,2}(z,s)\ \ \cdots\ \ V_{m,N_m}(z,s)]^T \tag{1}$$

$$\boldsymbol{I}_m(z,s) = [I_{m,1}(z,s)\ \ I_{m,2}(z,s)\ \ \cdots\ \ I_{m,N_m}(z,s)]^T \tag{2}$$

are given by

$$\frac{d\boldsymbol{V}_m(z,s)}{dz}+\boldsymbol{Z}_m(s)\boldsymbol{I}_m(z,s) = \boldsymbol{E}_{F,m}(z,s)+\boldsymbol{L}_m\boldsymbol{i}_m(z,0) \tag{3}$$

$$\frac{d\boldsymbol{I}_m(z,s)}{dz}+\boldsymbol{Y}_m(s)\boldsymbol{V}_m(z,s) = \boldsymbol{J}_{F,m}(z,s)+\boldsymbol{C}_m\boldsymbol{v}_m(z,0) \tag{4}$$

$$
\begin{aligned}
\boldsymbol{Z}_m(s) &= \boldsymbol{L}_m s + \boldsymbol{R}_m + \boldsymbol{B}_m \sqrt{s} &(5)\\
\boldsymbol{Y}_m(s) &= \boldsymbol{C}_m s + \boldsymbol{G}_m &(6)
\end{aligned}
$$

where $z$ denotes the position of the line and $s$ the Laplace operator[1],[2].

The per unit length $N_m \times N_m$ matrices are $\boldsymbol{L}_m$ (inductance), $\boldsymbol{C}_m$ (capacitance), $\boldsymbol{R}_m$ (resistance), $\boldsymbol{G}_m$ (leakage) and $\boldsymbol{B}_m$ (coefficient of skin effect loss). $N_m \times 1$ vectors $\boldsymbol{E}_{F,m}(z,s)$ and $\boldsymbol{J}_{F,m}(z,s)$ are generated by incident fields $\vec{\boldsymbol{E}}^{inc}$ and $\vec{\boldsymbol{H}}^{inc}$. $N_m \times 1$ vectors $\boldsymbol{i}_m(z,0)$ and $\boldsymbol{v}_m(z,0)$ are initial current and voltage distribution vectors in the time domain at $t=0$.

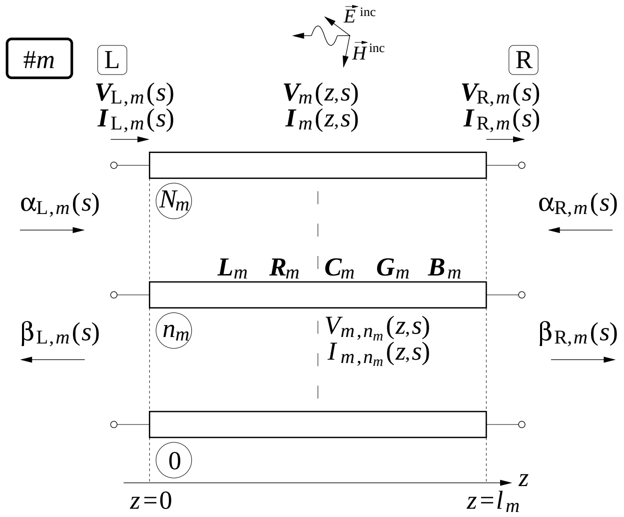


Fig.1 Multi-conductor uniform transmission line

## 2.2 General solutions for voltage and current vectors

General solution of the voltage vector at position $z$ becomes

$$
\begin{aligned}
\boldsymbol{V}(z,s) &= \exp[-\boldsymbol{Q}_m(s)z]\boldsymbol{K}_m^+(s)\\
&+ \exp[\boldsymbol{Q}_m(s)(z-l_m)]\boldsymbol{K}_m^-(s)\\
&+ \int_0^z \exp[-\boldsymbol{Q}_m(s)(z-\tau)]\boldsymbol{M}_m^+(\tau,s)d\tau\\
&+ \int_0^z \exp[\boldsymbol{Q}_m(s)(z-\tau)]\boldsymbol{M}_m^-(\tau,s)d\tau \quad (7)
\end{aligned}
$$

and the current vector becomes

$$
\begin{aligned}
\boldsymbol{I}(z,s) &= \boldsymbol{Z}_{0,m}^{-1}\left\{\exp[-\boldsymbol{Q}_m(s)z]\boldsymbol{K}_m^+(s)\right.\\
&- \left.\exp[\boldsymbol{Q}_m(s)(z-l_m)]\boldsymbol{K}_m^-(s)\right\}\\
&+ \boldsymbol{Z}_{0,m}^{-1}\left\{\int_0^z \exp[-\boldsymbol{Q}_m(s)(z-\tau)]\boldsymbol{M}_m^+(\tau,s)d\tau\right.\\
&- \left.\int_0^z \exp[\boldsymbol{Q}_m(s)(z-\tau)]\boldsymbol{M}_m^-(\tau,s)d\tau\right\} \quad (8)
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{Q}_m^2(s) &= \boldsymbol{Z}_m(s)\boldsymbol{Y}_m(s) &(9)\\
\boldsymbol{Z}_{0,m}(s) &= \boldsymbol{Q}_m(s)\boldsymbol{Y}_m^{-1}(s). &(10)
\end{aligned}
$$

We call $\boldsymbol{Q}_m(s)$ as the propagation matrix and $\boldsymbol{Z}_{0,m}(s)$ as the characteristic impedance matrix. $\boldsymbol{M}_m^+(z,s)$ and $\boldsymbol{M}_m^-(z,s)$ are vectors caused by the distributed sources and are given by

$$
\begin{aligned}
\boldsymbol{M}_m^+(z,s) &= \frac{1}{2}[\boldsymbol{E}_{F,m}(z,s) + \boldsymbol{L}_m \boldsymbol{i}_m(z,0)]\\
&+ \frac{1}{2}\boldsymbol{Z}_{0,m}(s)[\boldsymbol{J}_{F,m}(z,s) + \boldsymbol{C}_m \boldsymbol{v}_m(z,0)]
\end{aligned}
\quad (11)
$$

$$
\begin{aligned}
\boldsymbol{M}_m^-(z,s) &= \frac{1}{2}[\boldsymbol{E}_{F,m}(z,s) + \boldsymbol{L}_m \boldsymbol{i}_m(z,0)]\\
&- \frac{1}{2}\boldsymbol{Z}_{0,m}(s)[\boldsymbol{J}_{F,m}(z,s) + \boldsymbol{C}_m \boldsymbol{v}_m(z,0)].
\end{aligned}
\quad (12)
$$

$\boldsymbol{K}_m^+(s)$ and $\boldsymbol{K}_m^-(s)$ are constant vectors which do not depend on the position $z$ and determined by the conditions at both ends of the transmission line.

## 2.3 Formulation by Scattering matrix method

The exponential function of propagation matrix containes time forecasting operator in its elements. Since we do not realize time forecasting operator, it is inadequate to transform the expressions of eqns. (7) and (8) into the time domain. So we must use other formulations. For this purpose we define incident, reflect voltage wave vectors $\boldsymbol{\alpha}_{L,m}(s), \boldsymbol{\beta}_{L,m}(s)$ at near end $z=0$ and $\boldsymbol{\alpha}_{R,m}(s), \boldsymbol{\beta}_{R,m}(s)$ at far end $z=l_m$ as shown in Fig.1 by the following relations.

$$
\begin{aligned}
\boldsymbol{V}_{L,m}(s) &= \boldsymbol{\alpha}_{L,m}(s) + \boldsymbol{\beta}_{L,m}(s) &(13)\\
\boldsymbol{I}_{L,m}(s) &= \boldsymbol{Z}_{0,m}^{-1}(s)[\boldsymbol{\alpha}_{L,m}(s) - \boldsymbol{\beta}_{L,m}(s)] &(14)\\
\boldsymbol{V}_{R,m}(s) &= \boldsymbol{\alpha}_{R,m}(s) + \boldsymbol{\beta}_{R,m}(s) &(15)\\
\boldsymbol{I}_{R,m}(s) &= \boldsymbol{Z}_{0,m}^{-1}(s)[-\boldsymbol{\alpha}_{R,m}(s) + \boldsymbol{\beta}_{R,m}(s)] &(16)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{V}_{L,m}(s) &= \boldsymbol{V}_m(0,s) &(17)\\
\boldsymbol{I}_{L,m}(s) &= \boldsymbol{I}_m(0,s) &(18)\\
\boldsymbol{V}_{R,m}(s) &= \boldsymbol{V}_m(l_m,s) &(19)\\
\boldsymbol{I}_{R,m}(s) &= \boldsymbol{I}_m(l_m,s) &(20)
\end{aligned}
$$

By the above relations we obtain the following definitions.

$$
\boldsymbol{\alpha}_{L,m}(s) = \frac{1}{2}[\boldsymbol{V}_{L,m}(s) + \boldsymbol{Z}_{0,m}(s)\boldsymbol{I}_{L,m}(s)] \quad (21)
$$

$$\beta_{L,m}(s) = \frac{1}{2}[\boldsymbol{V}_{L,m}(s) - \boldsymbol{Z}_{0,m}(s)\boldsymbol{I}_{L,m}(s)] \quad (22)$$

$$\boldsymbol{\alpha}_{R,m}(s) = \frac{1}{2}[\boldsymbol{V}_{R,m}(s) - \boldsymbol{Z}_{0,m}(s)\boldsymbol{I}_{R,m}(s)] \quad (23)$$

$$\boldsymbol{\beta}_{R,m}(s) = \frac{1}{2}[\boldsymbol{V}_{R,m}(s) + \boldsymbol{Z}_{0,m}(s)\boldsymbol{I}_{R,m}(s)] \quad (24)$$

Eliminations of two vectors $\boldsymbol{K}_m^+(s)$ and $\boldsymbol{K}_m^-(s)$ yiels the formulation by the scattering matrix method.

$$\begin{bmatrix} \boldsymbol{\beta}_{L,m}(s) \\ \boldsymbol{\beta}_{R,m}(s) \end{bmatrix} = \boldsymbol{S}_m(s) \begin{bmatrix} \boldsymbol{\alpha}_{L,m}(s) \\ \boldsymbol{\alpha}_{R,m}(s) \end{bmatrix} + \boldsymbol{\Phi}_m(s) \quad (25)$$

The matrix $\boldsymbol{S}_m(s)$ defined by

$$\boldsymbol{S}_m(s) = \begin{bmatrix} \boldsymbol{0} & \exp[-\boldsymbol{Q}_m(s)l_m] \\ \exp[-\boldsymbol{Q}_m(s)l_m] & \boldsymbol{0} \end{bmatrix} \quad (26)$$

is called the scattering matrix and the vector

$$\boldsymbol{\Phi}_m(s) = \begin{bmatrix} -\int_0^{l_m} \exp[-\boldsymbol{Q}_m(s)\tau]\boldsymbol{M}_m^-(\tau, s)d\tau \\ \int_0^{l_m} \exp[-\boldsymbol{Q}_m(s)(l_m - \tau)]\boldsymbol{M}_m^+(\tau, s)d\tau \end{bmatrix} \quad (27)$$

is generated by the distributed sources. All the elements of matrix $\boldsymbol{S}_m(s)$ and vector $\boldsymbol{\Phi}_m(s)$ do not contain time forecasting operators and we can invert the relation of eq. (25) into the time domain.

# 3 Formulation of hybrid system in time domain

The overall equations governing the whole system are formulated in the time domain. By the discretization we solve the simultaneous equations numerically.

## 3.1 Formulation for distributed portion

For the distributed portion composed of interconnects we obtain

$$\begin{bmatrix} \boldsymbol{\beta}_{L,m}(t) \\ \boldsymbol{\beta}_{R,m}(t) \end{bmatrix} = \frac{d}{dt}\boldsymbol{s}_m^s(t) * \begin{bmatrix} \boldsymbol{\alpha}_{L,m}(t) \\ \boldsymbol{\alpha}_{R,m}(t) \end{bmatrix} + \boldsymbol{\phi}_m(t) \quad (28)$$

$$\boldsymbol{v}_{L,m}(t) = \frac{d}{dt}\boldsymbol{z}_{0,m}^s(t) * \boldsymbol{i}_{L,m}(t) + 2\boldsymbol{\beta}_{L,m}(t) \quad (29)$$

$$\boldsymbol{v}_{R,m}(t) = -\frac{d}{dt}\boldsymbol{z}_{0,m}^s(t) * \boldsymbol{i}_{,m}(t) + 2\boldsymbol{\beta}_{R,m}(t) \quad (30)$$

$$\boldsymbol{\alpha}_{L,m}(t) = \boldsymbol{v}_{L,m}(t) - \boldsymbol{\beta}_{L,m}(t) \quad (31)$$

$$\boldsymbol{\alpha}_{R,m}(t) = \boldsymbol{v}_{R,m}(t) - \boldsymbol{\beta}_{R,m}(t) \quad (32)$$

where

$$\boldsymbol{s}_m^s(t) = \mathcal{L}^{-1}\left[\frac{\boldsymbol{S}_m(s)}{s}\right] \quad (33)$$

$$\boldsymbol{z}_{0,m}^s(t) = \mathcal{L}^{-1}\left[\frac{\boldsymbol{Z}_{0,m}(s)}{s}\right] \quad (34)$$

and the asterisk $*$ denotes the convolution integral.

Since the $s$ functions $\boldsymbol{S}_m(s), \boldsymbol{Z}_{0,m}(s)$ and $\boldsymbol{\Phi}_m(s)$ are known, we can obtain $\boldsymbol{s}_m^s(t), \boldsymbol{z}_{0,m}^s(t)$ and $\boldsymbol{\phi}_m(t)$ as time sequences calculated in advannce by numerical Laplace inversion method[3].

## 3.2 Discretization of convolution integral

For the differential type convolution integral

$$\boldsymbol{h}(t) = \frac{d}{dt}\int_0^t \boldsymbol{f}(t-\tau)\boldsymbol{g}(\tau)d\tau \quad (35)$$

discretization by step size $\Delta t$ yields

$$\boldsymbol{h}(n) = \begin{cases} \boldsymbol{0} & n = 0 \\ \boldsymbol{f}(0)\boldsymbol{g}(n) & n = 1 \\ \sum_{k=1}^{n-1}[\boldsymbol{f}(n-k) - \boldsymbol{f}(n-k-1)]\boldsymbol{g}(k) + \boldsymbol{f}(0)\boldsymbol{g}(n) & 2 \le n \end{cases} \quad (36)$$

where $\boldsymbol{f}(n)$ means $\boldsymbol{f}(n\Delta t)$.

## 3.3 Formulation for lumped portion

For the lumped portion composed of linear/nonlinear elements and dependent sources and connected to the transmission lines at the both ends, we obtain the following nonlinear vector equation in the time domain.

$$\boldsymbol{F}_{NL}[\boldsymbol{x}(t), \boldsymbol{s}_{lump}(t), t] = \boldsymbol{0} \quad (37)$$

$$\boldsymbol{x}(t) = [\boldsymbol{v}_{L,m}^T(t)\ \boldsymbol{v}_{R,m}^T(t)\ \boldsymbol{i}_{L,m}^T(t)\ \boldsymbol{i}_{R,m}^T(t)]^T \quad (38)$$

and $\boldsymbol{s}_{lump}(t)$ is the vector composed of independent sources.

Eq. (37) is discretized and solved by Newton's method.

# 4 Numerical example

Let us consider a two phase loss less transmission line terminated by linear resistors and nonlinear diode as shown in Fig.2. Per unit length indactance and capacitance matrices of the line are given by

$$\boldsymbol{L} = \mu_0\epsilon_0\boldsymbol{C}^{-1} = \frac{\mu_0}{2\pi}\begin{bmatrix} \ln 100 & \ln 20 \\ \ln 20 & \ln 100 \end{bmatrix}. \quad (39)$$

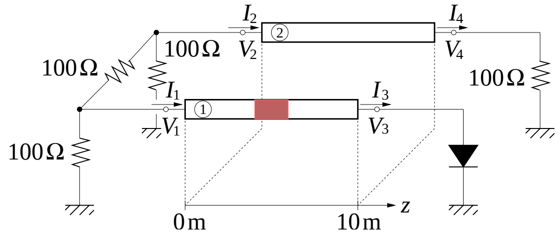


Fig.2 Two phase transmission line terminated by linear resistors and nonlinear diode.

The characteristic of nonlinear diode is given by

$$i_d(t) = \left\{ \exp\left[\frac{v_d(t)}{V_T}\right] - 1.0 \right\} \times 10^{-8} \quad \mathrm{A} \tag{40}$$

where $V_T = 25.0\mathrm{mV}$. Hatched portion of conductor 1 ($4m \leq z \leq 6m$) is immersed in half sinusoidal pulse incident electric field shown in Fig.3.

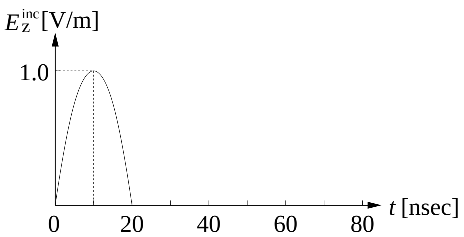


Fig.3 Waveform of incident electric field

Calculated transient voltages at near and far ends are shown in Fig.4 and Fig.5. All the obtained results agree well with those anticipated by the theoretical consideration.

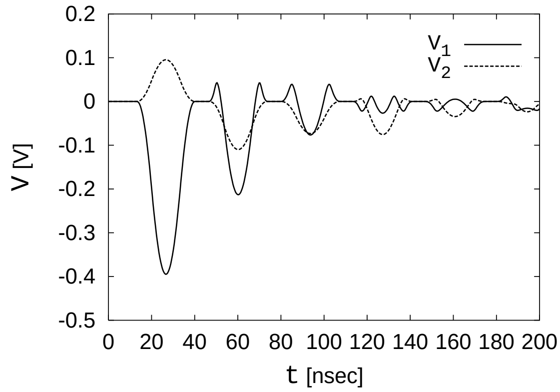


Fig.4 Transient voltage waveforms of $v_1$ and $v_2$

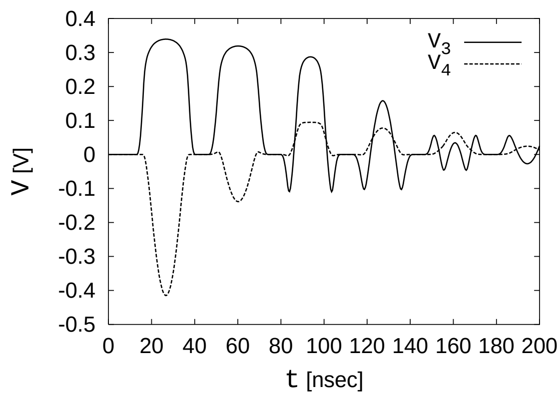


Fig.5 Transient voltage waveforms of $v_3$ and $v_4$

# 5 Conclusion

A computationally efficient method to calculate the transients in the hybrid system composed of linear/nonlinear lumpued elements and frequency dependent lossy interconnects is shown. For distributed portion, straightforward matrix formulation by scattering matrix method is derived by Laplace transformation. The overall equations governing the whole system are discretized and solved numerically in the time domain.

# References


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