Study of Existence and Structure of Chaotic Attractors

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Abstract

This paper introduces the concepts of space-bifurcation, singular saddle periodic orbit, hybrid attractor, synchronous orbit, combination orbit, and random bifurcation, and a theorem with three corollaries is presented. Then, a fractal layered structure of chaotic attractors is proposed, which is helpful for us to understand both the determinacy and randomness of chaotic motions.

1. Introduction

Generally speaking, the ability to obtain detailed knowledge of a chaotic attractor has undergone a tremendous advance in recent years [1-7], but the ability to integrate this knowledge has been stymied by the lack of a proper conceptual framework within which to describe chaos-producing mechanism and basic structure of the attractors [7]. This paper, based on the Reference[5], proposes a theoretical framework of periodic orbit analysis and provide practical techniques applicable to engineering problems.

2. Hybrid Attractor

Consider the three-order dissipative dynamical system described by the equations

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(x), \quad \boldsymbol{x} \in \Re^3 \tag{1}$$

where $\boldsymbol{f}: \Re^3 \to \Re^3$ is p -times differentiable ($p \geq 1$) with a continuous derivative, $\Re^3$ stands for the real space of dimension three, and $\boldsymbol{f}(\boldsymbol{x})$ are nonlinear functions.

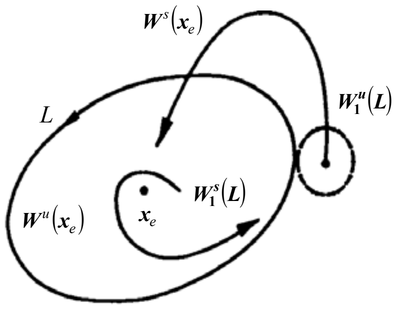

Fig. 1: Hybrid attractor $(\boldsymbol{x}_e, \boldsymbol{L})$ and circulating flow.

Fig.1 shows the sketch of a phase portrait for (1) . x_e is an EP for (1) if $f(x_e)=0$. The linearized system of (1) can be written as

$$\frac{d\boldsymbol{x}}{dt} = D\boldsymbol{f}(\boldsymbol{x}_e)\boldsymbol{x} \tag{2}$$

We call x_e for (1) a saddle focus (a semi-stable EP) if the eigenvalues of the 3×3 real matrix $D(x_e)$, the Jacobian derivative of $\boldsymbol{f}$ at x_e , are of the form

$$\gamma, \quad \sigma \pm j\omega_0, \quad \sigma\gamma < 0 \tag{3}$$

where γ, σ and ω_0 are real , $\omega_0 \neq 0$. Associated with x_e will be a 2-D eigenplane , $\boldsymbol{E}^c(\boldsymbol{x}_e)$ corresponding to the complex conjugate eigenvalues $\sigma \pm j\omega$ and 1-D eigenline $\boldsymbol{E}^r(\boldsymbol{x}_e)$ corresponding to the real eigenvalue γ . If

$$|\gamma| > |\sigma| > 0 \tag{4}$$

is satisfied, then (4) is called Shil'nikov inequality.

We call L a saddle periodic orbit (SPO), if it has a stable manifold $W^s(L)$ and an unstable manifold $W^u(L)$. In Fig.1, $W_1^u(L)$ and $W_1^s(L)$ denote the parts of $W^u(L)$ and $W^s(L)$, respectively. $W^s(\boldsymbol{x}_e)$ and $W^u(\boldsymbol{x}_e)$ are the stable and unstable manifolds of x_e , respectively. A segment of trajectory in the manifolds may be a direct-like motion or spiral-like motion . Suppose the trajectory segment can be represented by $\boldsymbol{x} = \boldsymbol{x}_e + \tilde{\boldsymbol{x}}_d(t) + \tilde{\boldsymbol{x}}_s(t)$, according to the notion of time-frequency analysis [8] , here $\tilde{x}_d(t)$ and $\tilde{x}_s(t)$ are the average and alternating components of $\boldsymbol{x}$, respectively. If $|\tilde{x}_d(t)| >> |\tilde{x}_s(t)|$, $\boldsymbol{x}$ is called a direct-like motion (direct motion or DM, for short) . If $|\tilde{x}_d(t)| << |\tilde{x}_s(t)|$, $\boldsymbol{x}$ is called a spiral-like motion (spiral motion or SM, for short). Here the symbol $|\ |$ is Euclidean norm. A SPO is called a singular SPO (SSPO) if its local stable or unstable manifold in its neighborhood is direct motions. If a SSPO is a SPO with shortest period, we call such a SSPO a main SSPO. A saddle EP and a main SSPO constitute a hybrid attractor if the directions of the manifolds are consistent with

$$W^u(\boldsymbol{x}_e) = W_1^s(L) \,,\, W^s(\boldsymbol{x}_e) = W_1^u(L) \tag{6}$$

A typical regime of (1) can be described with the aid of Fig.1. The hybrid attractor (x_e, L) forms such a complex vector field that (6) is satisfied, and other parts of $W^u(L)$ go through the neighborhood of L and the subspace near both x_e and L , respectively, and become spiral motions, the parts of $W^s(L)$ except $W_1^s(L)$. This means that there may exist a circulating flow in the neighborhood of the hybrid attractor.

3. Instability of Saddle Periodic Orbit

In existing literature, the theory of limit cycle stability develops from the so-called linearized variational equations [9], and various methods were proposed to reduce variational systems based on differential equations with periodic coefficients to those based on the differential equations with constant coefficients [9,10]. Now this approach is extended to chaotic systems.

To find an approach to the analysis of stability of a SPO (may be SSPO) x_l , we should consider the trajectory segments of the perturbed system [9] in the subspace around the space-bifurcation region (see the definition given in next section). The perturbed trajectory segments can be used to yield nonlinear variational equations which have the form

$$\frac{d\mathbf{y}}{dt} = \varphi\ (\mathbf{y}), \quad \mathbf{y} \in \Re^3 \tag{7}$$

where the norm of the c^1 – perturbation, $\mathbf{y} = \mathbf{x} - \mathbf{x}_l$, and its first derivative are not very small in an sufficiently large neighborhood of the SPO. An equivalent linearization approximation (e.g. the averaging of linear approximations in smaller regions) of the dynamics of (7) can be described by a set of linear equations in the form

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y} \tag{8}$$

where

$$A = \begin{cases} A_1 & \text{(in the attracting subspace of SPO)} \\ A_2 & \text{(in the repelling subspace of SPO)} \end{cases}$$

and A_i is a 3×3 matrix .The eigenvalues of A_i are of the form ,respectively,

$$\beta_1, \quad \alpha_1 \pm j\omega \tag{9}$$

and

$$\beta_2, \quad \alpha_2 \pm j\omega \tag{10}$$

where β_i , α_i and ω are real , $\omega \approx \omega_0$ is the fundamental frequency of x_l . (9) and (10) can be used for verification of instability of SPO. If the inequalities

$$\beta_1, \ \alpha_1 < 0, \ \beta_2 > 0, \ \alpha_2 < 0 \tag{11}$$

are satisfied , then the SPO is a SSPO.

Remark Various applications of line approximation of chaotic dynamics can be found in the existing literature [4].

4. Space-Bifurcation of Orbit

We now examine the typical theoretical chaotic regime of (1) with the aid of Fig.1. The hybrid attractor $(\mathbf{x}_e, L)$ lies on a plane S (not shown in Fig.1) , $E^c(\mathbf{x}_e) \subset S$, and $U_b \subset \Re^3$ is a neighborhood of L that contains a segment of L . S is a basin of L , D_1 and D_2 denote the subspaces in the upper and lower sides of S , respectively. There exists a spiral motion $W^s(L)$ on S and a direct motion (DM) $W^u(L)$ in D_1 . A trajectory $\mathbf{x}_{rm}$ is called a recurving motion if (6) is satisfied in some subspaces and $\mathbf{x}_{rm}$ consists of an orbit segment $\mathbf{x}_s$ in $W^s(L)$ and other one $\mathbf{x}_d$ in $W^u(L)$ .Here the word recurving means that it is generally not ε – neighborhood recurrence. Furthermore, if a chaotic attractor is in the neighborhood of *n* saddle focuses, then the attractor is called *n* - core chaotic attractor. If a chaotic attractor is in the neighborhood of *n* hybrid attractors , then the attractor is called *n* -spiral chaotic attractor.

The one-core chaotic attractor described by Fig.1 can be thought to be in a subspace (main subspace) that contains all the spiral orbits and another subspace that contains all the DM orbits. Thus, the chaotic attractor consists of many recurving orbits, and there exists a countable sequence t_n (the integers $n \in (-\infty, +\infty)$) of the times at which *n*th recurving orbit ($\mathbf{x}_{sn}$ and $\mathbf{x}_{dn}$) starts to appear. The values $T_n = t_{n+1} - t_n$ are called recurving times. The dividing point of orbit segments $\mathbf{x}_{sn}$ and $\mathbf{x}_{dn}$ near L can be located by inspecting ρ_L , where $\rho_L = min\left\{ \left|\mathbf{x} - \mathbf{x}_L\right| \right\}$ for $\mathbf{x}_L \in L$. The dividing point is defined as the point where ρ_L is minimum. Note that $d\rho_l/dt$ has a sharp tuning point near the dividing point, and this can be used for computation of the dividing point. Likewise, another dividing point, one between $\mathbf{x}_{dn}$ and $\mathbf{x}_{s(n+1)}$, corresponds to the minimum value of ρ_e , where $\rho_e = \left|\mathbf{x} - \mathbf{x}_e\right|$.

The dividing points are important for some problems in engineering. The dividing points of $\mathbf{x}_s$ and $\mathbf{x}_d$ near the limit cycle are called space-bifurcation points, and they form a space-bifurcation region. A moving phase point passing a space-bifurcation point will switch randomly, under perturbation, between a trajectory on S and one in D_1. This causes the random bifurcation mentioned later.

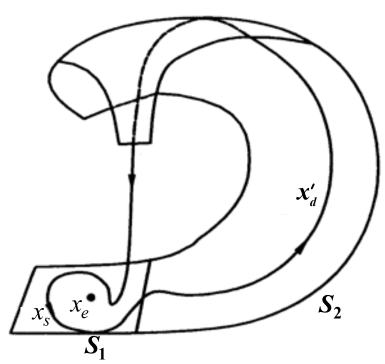

Fig.2: A saddle periodic orbit consists of a spiral trajectory segment $\mathbf{x}_s$ and a direct motion trajectory segment $\mathbf{x}_d'$

As is shown in Fig.2, the SPO (or SSPO) may be on a surface that consists of a plane S_1 and a surface S_2 . The geometric construction is the same as the Fig.4 in the literature [7]. $\mathbf{x}_s$ is a segment of spiral orbit that attracts all the spiral orbits on S_1 . And $\mathbf{x}'_d$ on S_2 is a direct-motion-like orbit and attracts all the phase points in its neighborhood on S_2 . It is noted that in this case all the direct motion orbits except $\mathbf{x}'_d$ in the neighborhood of the SSPO are outside S_1 and S_2 . Hence , for the property of attracting and repelling , there is no important difference between the SSPO in Fig.1 and that in Fig.2.

From Fig.1 we see that L attracts all the trajectories in $W^s(L)$, and must attract a segment of each SPO with longer period. So, L is called a main SPO. It is obvious, only a single-cycle SPO or a SPO with the shortest period can be main SPO. It is noted that (1) may have no other short period orbits. For example, a system has no one-cycle SPO but it has a two-cycle SPO and others, then the two-cycle SPO is the main SPO. Hence, a Shil'nikov homoclinic loop [1] of the shortest period can be a main SPO. Obviously, a main SSPO is generally the main SPO. Based on Fig.1 and related descriptions, a qualitative model of one-core chaotic attractor can be given as follows: If the solution x of (1) is bounded and there exists only one hybrid attractor, for which the associated eigenvalues of x_e satisfy the Shil'nikov inequality (4), and L is a main SSPO , then (1) exhibits a chaotic attractor . This model is a simplified description of the theorem 1 given in next section.

We will see later that the model can be extended to two-core and 2-spiral chaotic attractors. Furthermore , it is conceivable that the model may be extended to the case that the main SSPO is replaced by a subharmonic oscillation(see the Reference [11]).

5. Theorem of Existence of Chaotic Attractors

Consider the system (1).

Definition 1 An attractor is strange if it is neither an equilibrium point nor a torus attractor. A strange attractor may be chaotic or non-chaotic[12].

Definition 2 An attractor is chaotic if it is such an attractor on which motions are deterministic nonperiodic flows with sensitive dependence on initial conditions.

Hypothesis 1 There exists a basin A for (1) in which there is no stable n-dimensional torus attractor (n=1 , 2 , 3) .

Theorem 1 Given the nonlinear system in (1). Let x_e and L be an EP and a single-cycle SPO for (1), respectively, which lie on a plane S . If system (1) satisfies Hypothesis 1 , and x_e and L are in basin A , then system (1) exhibits a chaotic attractor if the following conditions are satisfied :

(i) The origin of (1) is x_e , or is unstable. x_e is a saddle focus whose characteristic eigenvalues in (3) satisfy the Shil'nikov inequality (4).

(ii) S is a basin of L , and L is a main SPO whose associated eigenvalues satisfy:

$$\begin{aligned} &\beta_1,\ \alpha_1<0;\ \ \beta_2>0;\ \ \alpha_2<0 \\ &\beta_2\gamma<0;\ \ \alpha_1\sigma<0 \end{aligned} \tag{12}$$

Moreover, if system (1) does not satisfy Hypothesis 1 but does satisfy the above conditions, then system (1) exhibits transient chaos .

Proof (Omitted here)

Corollary 1 Consider the system described by(1). Suppose the satisfaction of Hypothesis 1 is unknown or not considered. If all conditions stated in Theorem 1 are satisfied, then system (1) exhibits either a chaotic attractor or transient chaos.

Based on the theorem, Corollaries 2 and 3 can be presented [11], which are on the existences of two-core and 2-spiral chaotic attractors, respectively.

6. Synchronous Orbits in Chaos

According to the qualitative model of one-core chaotic attractor and related descriptions, a comparatively simple chaotic motion includes two types of components, the spiral motions with the period $T=2\pi/\omega$ and the direct motions with the recurving times $T_n=2\pi/\Omega_n$, and Ω_n may be thought as a time-varing frequency from the view point of time-frequency analysis [8]. This is similar to a system that consists of two coupled oscillators, of which the fundamental angular frequencies are Ω and ω , respectively. If Ω_n is constant and ω/Ω_n is rational, then the motion is periodic (the two oscillations synchronize each other, i.e. they are in frequency locking [13]), and a periodic orbit appears in the corresponding phase space. Hence, we may say that all the periodic orbits within a chaotic attractor are synchronous orbits. In other words, each periodic orbit within the attractor is corresponding to a synchronous regime of the two components. As is described in the literature [13], the above description of the two-oscillations system allow us to consider the standard circle map and then use Arnol'd tongue diagram [13]. The diagram indicates that there exist two classes of synchronous orbits, of which the period relations satisfy, respectively.

$$T_n=mT \quad (m\in \text{positive integers}) \tag{13}$$

and

$$T_n=\frac{m}{k}T \tag{14}$$

$$(m,\ k\in \text{positive integers, coprime; } k\neq 1)$$

The two classes of orbits are called period-m orbits (subharmonic oscillations) and period-k(*m/k*) orbits (super-subharmonic oscillations), respectively. As is mentioned above, they are unstable for a chaotic system.

We have found that there exists another class of synchronous orbits with period relation

$$\sum_{n=1}^{k}T_n=T\sum_{n=1}^{k}m_n \tag{15}$$

where m_n are generally not integers, but $M_n=\Sigma\ m_n(n=1,2,\ldots k)$ is a positive integer. Every periodic orbit of this class consists of two or more pieces, and each of the pieces can be regarded as a part of a periodic orbit with period M_nT, as is shown in Fig.3. Thus, they are combination orbits, and may be called period-M_n orbits. We see from (14) and (15) that the period-*k(m/k)* orbits are a special class of the period- M_n orbits. On the other hand, we see from Fig.3 that the shape of trajectory of a period- M_n orbit is much different from that of a period-m orbit with m=M_n . For example, the shapes of trajectories of the period-(2.2+2.8) and period-5 are different. It is worthy to point out that the existence of period-M_n orbits is the most important cause among those of existence of infinite periodic orbits within a real chaotic attractor, because the number of "hot" period-m orbits is limited.

7. Layered Structure

From the proof of the theorem 1 we see that a trajectory in the neighborhood of a SSPO within a chaotic attractor is a non-periodic orbit with sensitive dependence on initial conditions, which forms a part of the chaotic subattractor in the neighborhood, or is a orbit that stays in the neighborhood for less than one recurving time and then goes into the neighborhood of another SSPO. This concept and above description of synchronous orbits allow us to image that a chaotic attractor consists of chaotic subattractors, and has an infinitely layered structure.

Fig.4 shows a sketch of construction of periodic orbits within a one-core chaotic attractor. Like a practical phase portrait, the main periodic orbit and spiral trajectory are not on the associated eigenplane of x_e . The dividing points of the spiral orbits and DM orbits near the main SSPO are the space-bifurcation points, and they form a space-bifurcation region. The dividing point of spiral orbit and DM orbit of a trajectory going by x_e is the nearest point to x_e . The spiral trajectory shown in Fig.4 is on a cone-like surface that corresponds to the plane in Fig.1. Similarly, each spiral trajectory of other period-m orbits is on its own cone-like surface. Thus, the periodic orbits form an infinitely layered structure. If the spiral orbit segments of a SSPO are distributed over n cone-like surfaces, then the SSPO is called a level-n SSPO. If the main SSPO becomes a torus attractor that corresponds to a subharmonic or quasi-periodic motion, then the spiral orbit attracted by the torus is on a “spiral tube”, and the cone-like surface turns out to be a cone-like “spiral tube”.

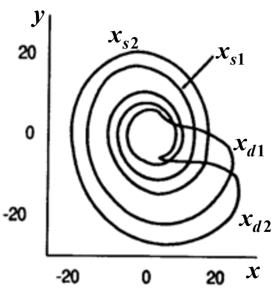

Fig.3: A periodic combination orbit

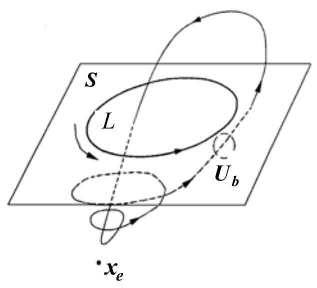

Fig.4: A sketch used to describe that the spiral trajectory segments of non-main SSPO of a system are on layered cone-like surfaces

As is mentioned above, the random switching of a moving point occurs under perturbations. A phase point evolving in the neighborhood of a spiral trajectory segment of a synchronous orbit will leap upwards when passing through the space-bifurcation region, and goes randomly into one of the neighborhoods of the periodic orbits, and thus forms a segment of chaotic trajectory. The similar processes occur consequently, and a chaotic attractor appears.

Similarly, the layered structure exists in the small neighborhood of each of the period-m and period-M_n orbits. By thinking the proof of theorem 1 and related descriptions, we can see that there is the similarity between the main SSPO and a non-main SSPO. As is mentioned above, period-2 orbit may be main SPO. In this case, the shapes of the main and non-main SPO’s are of same type. Every non-main SPO goes through the space-bifurcation region, and has its own smaller associated space-bifurcation region, through which all the trajectories attracted by the non-main SPO will go. And it has an attracting region in its small neighborhood. So, the non-main SSPO can be thought as special “main SSPO”. Using the concepts in the proof of theorem 1 to the special “main SSPO” and x_e , we can see that there exist synchronous orbits, layered structure and chaotic behavior in the small neighborhood of each non-main SSPO in the chaotic attractor. Thus, the layered structure of a chaotic attractor is a fractal structure, and there exist infinite SSPO’s in the attractor.

The process of generating a limit cycle from a fixed point is called a Hopf bifurcation. For a comparison, the process of generating a level-n SSPO from a level-n’ SSPO with $n' \neq n$ may be called a random bifurcation (The process of generating a level-n periodic-M_n orbit from a level-n periodic-M_n’ orbit with $M_n' \neq M_n$ may be regarded as a special case of random bifurcation). But the two kinds of bifurcation are much different. The random bifurcation occurs randomly via the space-bifurcation, and we can not predict when it occurs and which SSPO will appear.

8. Conclusion

This paper has presented the theorem with its three corollaries. The theorem gives a chaos-producing mechanism: hybrid attractors are the root cause of attractor that exists in the neighborhood of the attractors. The theorem is applicable to a number of important problems(see the Reference [11]).

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