Balanced $C_8$-Bowtie Decomposition Algorithm of Complete Graphs

Kazuhiko Ushio  Hideaki Fujimoto
Department of Informatics  Department of Electric and Electronic Engineering
Faculty of Science and Technology
Kinki University
ushio@is.kindai.ac.jp  fujimoto@ele.kindai.ac.jp

1. Introduction

Let $K_n$ denote the complete graph of $n$ vertices. Let $C_8$ be the cycle on 8 vertices. The $C_8$-bowtie (or the $C_8$-2-windmill) is a graph of 2 edge-disjoint $C_8$’s with a common vertex and the common vertex is called the center of the $C_8$-bowtie. When $K_n$ is decomposed into edge-disjoint sum of $C_8$-bowties, it is called that $K_n$ has a $C_8$-bowtie decomposition. Moreover, when every vertex of $K_n$ appears in the same number of $C_8$-bowties, it is called that $K_n$ has a balanced $C_8$-bowtie decomposition and this number is called the replication number.

In this paper, it is shown that the necessary and sufficient condition for the existence of a balanced $C_8$-bowtie decomposition of $K_n$ is $n \equiv 1 \pmod{32}$.

It is a well-known result that $K_n$ has a $C_3$-decomposition if and only if $n \equiv 1 \text{ or } 3 \pmod{6}$. This decomposition is known as a Steiner triple system. See Colbourn and Rosa[1] and Wallis[5, Chapter 12 : Triple Systems]. Hor´ak and Rosa[2] proved that $K_n$ has a $C_3$-bowtie decomposition if and only if $n \equiv 1 \text{ or } 9 \pmod{12}$. This decomposition is known as a $C_3$-bowtie system. In this sense, our balanced $C_8$-bowtie decomposition of $K_n$ is to be known as a balanced $C_8$-bowtie system.

2. Balanced $C_8$-bowtie decomposition of $K_n$

We use the following notation for a $C_8$-bowtie.

Notation. We denote a $C_8$-bowtie passing through $v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 - v_1 - v_9 - v_{10} - v_{11} - v_{12} - v_{13} - v_{14} - v_{15} - v_1$ by $\{(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8), (v_1, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15})\}$.

We have the following theorem.

Theorem. $K_n$ has a balanced $C_8$-bowtie decomposition if and only if $n \equiv 1 \pmod{32}$.

Proof. (Necessity) Suppose that $K_n$ has a balanced $C_8$-bowtie decomposition. Let $b$ be the number of $C_8$-bowties and $r$ be the replication number. Then $b = n(n-1)/2$ and $r = 15(n-1)/32$. Among $r$ $C_8$-bowties having a vertex $v$ of $K_n$, let $r_1$ and $r_2$ be the numbers of $C_8$-bowties in which $v$ is the center and $v$ is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to $v$, $4r_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/2$ and $r_2 = 7(n-1)/16$. Therefore, $n \equiv 1 \pmod{32}$ is necessary.

(Sufficiency) Put $n = 32t + 1$. Construct $tn$ $C_8$-bowties as follows:

$B_1(t) = \{(i, i + 1, i + 4t + 2, i + 24t + 2, i + 14t + 2, i + 26t + 2, i + 8t + 2, i + 2t + 1), (i, i + 2, i + 4t + 4, i + 24t + 3, i + 14t + 4, i + 26t + 3, i + 8t + 4, i + 2t + 2)\}$

$B_2(t) = \{(i, i + 3, i + 4t + 6, i + 24t + 4, i + 14t + 6, i + 26t + 4, i + 8t + 6, i + 2t + 3), (i, i + 4, i + 4t + 8, i + 24t + 5, i + 14t + 8, i + 26t + 5, i + 8t + 8, i + 2t + 4)\}$

$\ldots$

$B_t(t) = \{(i, i + 2t - 1, i + 8t - 2, i + 26t, i + 18t - 2, i + 28t, i + 12t - 2, i + 4t - 1), (i, i + 2t, i + 8t, i + 26t + 1, i + 18t, i + 28t + 1, i + 12t, i + 4t)\}$

$(i = 1, 2, \ldots, n)$,

where the additions $i + x$ are taken modulo $n$ with residues $1, 2, \ldots, n$.

Then they comprise a balanced $C_8$-bowtie decomposition of $K_n$. 

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This complete the proof.

**Note.** We consider the vertex set $V$ of $K_n$ as $V = \{1, 2, \ldots, n\}$. The additions $i + x$ are taken modulo $n$ with residues $1, 2, \ldots, n$.

**Example 1. Balanced $C_8$-bowtie decomposition of $K_{33}$.**

$$B_i = \{(i, i+1, i+6, i+26, i+16, i+28, i+10, i+3),
(i, i+2, i+8, i+27, i+18, i+29, i+12, i+4)\} (i = 1, 2, \ldots, 33).$$

**Example 2. Balanced $C_8$-bowtie decomposition of $K_{65}$.**

$$B_i^{(1)} = \{(i, i+1, i+10, i+50, i+30, i+54, i+18, i+5),
(i, i+2, i+12, i+51, i+32, i+55, i+20, i+6)\}$$
$$B_i^{(2)} = \{(i, i+3, i+14, i+52, i+34, i+56, i+22, i+7),
(i, i+4, i+16, i+53, i+36, i+57, i+24, i+8)\} (i = 1, 2, \ldots, 65).$$

**Example 3. Balanced $C_8$-bowtie decomposition of $K_{97}$.**

$$B_i^{(3)} = \{(i, i+1, i+14, i+74, i+44, i+80, i+26, i+7),
(i, i+2, i+16, i+75, i+46, i+81, i+28, i+8)\}$$
$$B_i^{(2)} = \{(i, i+3, i+18, i+76, i+48, i+82, i+30, i+9),
(i, i+4, i+20, i+77, i+50, i+83, i+32, i+10)\}$$
$$B_i^{(3)} = \{(i, i+5, i+22, i+78, i+52, i+84, i+34, i+11),
(i, i+6, i+24, i+79, i+54, i+85, i+36, i+12)\} (i = 1, 2, \ldots, 97).$$

**Example 4. Balanced $C_8$-bowtie decomposition of $K_{129}$.**

$$B_i^{(3)} = \{(i, i+1, i+18, i+98, i+58, i+106, i+34, i+9),
(i, i+2, i+20, i+99, i+60, i+107, i+36, i+10)\}$$
$$B_i^{(2)} = \{(i, i+3, i+22, i+100, i+62, i+108, i+38, i+11),
(i, i+4, i+24, i+101, i+64, i+109, i+40, i+12)\}$$

**Example 5. Balanced $C_8$-bowtie decomposition of $K_{161}$.**

$$B_i^{(1)} = \{(i, i+1, i+22, i+122, i+72, i+132, i+42, i+11),
(i, i+2, i+24, i+123, i+74, i+133, i+44, i+12)\}$$
$$B_i^{(2)} = \{(i, i+3, i+26, i+124, i+76, i+134, i+46, i+13),
(i, i+4, i+28, i+125, i+78, i+135, i+48, i+14)\}$$
$$B_i^{(3)} = \{(i, i+5, i+30, i+126, i+80, i+136, i+50, i+15),
(i, i+6, i+32, i+127, i+82, i+137, i+52, i+16)\}$$
$$B_i^{(4)} = \{(i, i+7, i+34, i+128, i+84, i+138, i+54, i+17),
(i, i+8, i+36, i+129, i+86, i+139, i+56, i+18)\}$$
$$B_i^{(5)} = \{(i, i+9, i+38, i+130, i+88, i+140, i+58, i+19),
(i, i+10, i+40, i+131, i+90, i+141, i+60, i+20)\} (i = 1, 2, \ldots, 161).$$

**References**