Quantum Erasure Correction and Quantum Local Recovery

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Messages are encoded into a codeword

Message: "I enjoy Turkish foods."

$\Downarrow$ is **encoded** into

Codeword:

The purpose of encoding (in this talk) is to protect messages from errors (and erasures).
The topic of this talk is error correction of quantum information. The theory of error correction is called **coding theory**.
Your smartphone camera can reconstruct the message in this and the next pages.

Codeword with errors can be decoded

Your smartphone camera reconstructs the message,
even if errors corrupt a codeword (like below)

In this example, the positions of errors can be identified.

Erasures in coding theory

Erasure = an error with a known position in a codeword.

A code can correct twice as many erasures as it can correct errors. The same applies to quantum error correction.

A recent trend in fault-tolerant quantum computation is erasure correction, as positions of errors can be known on some quantum devices. The following papers explain how quantum erasures happen on some devices:

- Y. Wu, S. Kolkowitz, S. Puri, and J. D. Thompson. Erasure conversion for fault-tolerant quantum computing in alkaline earth Rydberg atom arrays. *Nature Communications*, 13:4657, 2022.
- M. Kang, W. C. Campbell, and K. R. Brown. Quantum error correction with metastable states of trapped ions using erasure conversion. *PRX Quantum*, 4:020358, 2023.

An $[[n,k]]_p$ QECC

p: a prime integer.
A quantum symbol of dimension p is called a **qudit** $\mathcal{H}_p = \mathbf{C}^p$: state space of a qudit
$|\psi\rangle \in \mathcal{H}_p^{\otimes k}$: quantum state of **message** consisting of k qudits
$Q \subset \mathcal{H}_p^{\otimes n}$: $[[n,k]]_p$ QECC (quantum error-correcting code)

1. A message $|\psi\rangle \in \mathcal{H}_p^{\otimes k}$ is **encoded** to a **codeword** $|\varphi\rangle \in Q \subset \mathcal{H}_p^{\otimes n}$ consisting of n qudits.
2. The codeword $|\varphi\rangle$ is sent and a density matrix ρ of n qudits is received.
3. The received state ρ is **decoded** to a (hopefully original) codeword $|\varphi'\rangle$ or to a message $|\psi'\rangle$.
4. Decoding is considered a success if the original is restored, and a failure otherwise. Alternatively, decoding is evaluated by the quantum fidelity whose value lies between 0 and 1.

Quantum errors and erasures I

$\{|0\rangle, |1\rangle, \ldots, |p-1\rangle\}$: an orthonormal basis of $\mathcal{H}_p$
X: $\mathcal{H}_p \to \mathcal{H}_p$, $|i\rangle \mapsto |(1+i) \bmod p\rangle$
Z: $\mathcal{H}_p \to \mathcal{H}_p$, $|i\rangle \mapsto \omega^i |i\rangle$
$\omega = \exp(2\pi\sqrt{-1}/p) \in \mathbf{C}$, $\mathbf{F}_p = \{0, 1, \ldots, p-1\}$: finite field with p elements
For $\vec{x} = (a_1, \ldots, a_n | b_1, \ldots, b_n)$, let

$$\begin{aligned} M(\vec{x}) &= X^{a_1} Z^{b_1} \otimes \cdots \otimes X^{a_n} Z^{b_n} \\ E_n &= \{\omega^i M(\vec{x}) : \vec{x} \in \mathbf{F}_p^{2n}, i \in \mathbf{F}_p\} \\ |E_n| &= p^{2n+1} \end{aligned}$$

For $p = 2$, they are the Pauli matrices:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Quantum errors and erasures II

- There are infinitely many unitary matrices on $\mathcal{H}_p^{\otimes n}$, but we have to consider only errors in E_n for successful error correction.
- Since scalar multiplication does not change the physical meaning of quantum states, quantum errors on n qudits can be identified with vectors in $\mathbf{F}_p^{2n}$.
- When $|\varphi\rangle$ is sent, the received state is assumed to be $U|\varphi\rangle$ for some $U \in E_n$.
- Quantum erasures are errors with known positions.

(r, δ) locality of a QECC

A QECC (or classical ECC) is said to have locality (r, δ) if

- For any coordinate $1 \leq i \leq n$,
- there exists a set of coordinates $J \subseteq \{1, \ldots, n\}$ of size $|J| \leq r + \delta - 1$ with $i \in J$
- such that any $\delta - 1$ erasures in J can be corrected by only manipulating qudits of a received state in J.

Meaning:
$\delta - 1$: the number of correctable erasures
r: the number of additional qudits needed to correct $\delta - 1$ erasures

A formal definition of (r, δ) locality of QECCs can be found in [1].

Historical developments of locality concepts of codes

1. Gopalan et al. [3] proposed the r-locality for classical ECCs, corresponding to $\delta = 2$ in (r, δ) locality, for a distributed storage network,
2. Prakash et al. [7] extended it to (r, δ) locality for classical ECCs,
3. Golowich and Guruswami [2] proposed the r-locality for QECCs, and
4. Galindo et al. [1] extended it to (r, δ) locality for QECCs.

In what follows, the relevance of locality of QECCs to quantum computers is reviewed.

Popular implementations of quantum computers

This list is not exhaustive.

- Superconductors
- Neutral atoms
- Trapped ions

Superconductors

Reference: [4] "Advancements in superconducting quantum computing".

- Considered most promising, and quantum computers on popular media are often of this type.
- Made by existing semiconductor fabrication techniques.
- Requires cooling to an absolute temperature of around a few kelvin.
- Qubits are placed and fixed on a wafer and their movements are usually impossible.
- Errors can be converted to erasures in some cases [6, 9].

Neutral atoms

Reference: [10] "Neutral atom quantum computing hardware: performance and end-user perspective".

- Neutral atoms are placed in a vacuum cell.
- Placement can be two- or three-dimensional.
- Can support many ($\simeq 1,000$) qubits.
- Qubits can be moved (with some noise) [10, Section 3.1.6], and selected qubits can be moved closer to allow local operations on them.
- Errors can be converted to erasures in some cases [11].

Trapped ions

Reference: [8] "Ion-based quantum computing hardware: performance and end-user perspective".

- Ions are trapped in a vacuum chamber.
- Gate fidelity is high.
- Can support a moderate number of ($\simeq 100$) qubits.
- Placement can be two-dimensional.
- Can operate at room temperature.
- Qubits can be moved (with some noise) [8, Section 3.2.2], and selected qubits can be moved closer to allow local operations on them.
- Errors can be converted to erasures in some cases [5].

Why locality matters.

- Quantum erasure correction definitely deserves investigation.
- Not every set of multiple qubits is manipulated easily. On superconducting devices, only neighboring qubits are usually manipulated together. Other quantum computers allow moving qubits, but it has some cost.
- Therefore, in quantum erasure correction, the number of manipulated qubits should be as small as possible, which is exactly the study of locality.

Relation between QECCs and classical ECCs I

For $\vec{x} = (x_1, \ldots, x_n)$, $\vec{y} = (y_1, \ldots, y_n) \in \mathbf{F}_{p^2}^n$,

$$\begin{aligned} \vec{x}^p &= (x_1^p, \ldots, x_n^p) \in \mathbf{F}_{p^2}^n \\ \langle \vec{x}, \vec{y} \rangle_e &= x_1 y_1 + \cdots + x_n y_n, \\ \langle \vec{x}, \vec{y} \rangle_h &= x_1 y_1^p + \cdots + x_n y_n^p = \langle \vec{x}, \vec{y}^p \rangle_e. \end{aligned}$$

For $C \subseteq \mathbf{F}_{p^2}^n$,

$$\begin{aligned} C^{\perp e} &= \{\vec{x} \in \mathbf{F}_{p^2}^n : \forall \vec{y} \in C, \langle \vec{x}, \vec{y} \rangle_e = 0\}, \\ C^{\perp h} &= \{\vec{x} \in \mathbf{F}_{p^2}^n : \forall \vec{y} \in C, \langle \vec{x}, \vec{y} \rangle_h = 0\}. \end{aligned}$$

Relation between QECCs and classical ECCs II [1]

CSS Construction

From $C \subseteq \mathbf{F}_p^n$ with locality (r, δ) such that $C^{\perp e} \subset C$, we can construct an $[[n, 2\dim C - n]]_p$ QECC with locality (r, δ).

Hermitian Construction

From $C \subseteq \mathbf{F}_{p^2}^n$ with locality (r, δ) such that $C^{\perp h} \subset C$, we can construct an $[[n, 2\dim C - n]]_p$ QECC with locality (r, δ).

By the above theorems, the construction of QECCs with locality is reduced to that of classical ECCs with locality and self-orthogonality. Several constructions have been proposed since quantum local recovery was proposed in 2023.

Localized error and erasure correction (classical case)

$\mathbf{F}_p^n \supset C$: classical ECC with locality (r, δ).
The i-th component has an erasure. $J \subset \{1, \ldots, n\}$, where $i \in J$, is a recovery set of size $r + \delta - 1$. Erasure indices belong to $I \subset J$. $\mathbf{F}_p^n \ni \vec{r}$ is a received word with t errors and e erasures in J, assuming $2t + e < \delta$.

1. Consider components of $\vec{r}$ in $J \setminus I$, which may have errors. Correct those errors in $J \setminus I$ by an error correction procedure of C only using components in $J \setminus I$.
2. Now components of $\vec{r}$ in $J \setminus I$ are error-free.
3. Erased components in I are reconstructed from those in $J \setminus I$ by solving a system of linear equations defined by C.

- The above procedure uses components only in J.
- Quantum case is almost the same but has additional issues (next page).

Localized error and erasure correction (quantum case) I

$\mathbf{F}_p^n \supset C$: classical ECC with locality (r, δ).
$C^{\perp e} \subset C$ is assumed (the CSS construction of QECCs)
$C^{\perp e}$ defines quantum measurements in decoders.
If $\{\vec{x}_1, \ldots, \vec{x}_\ell\}$ is a basis of $C^{\perp e}$, a pair of the observables defined by each $\vec{x}_j$ is measured.
For $L \subset \{1, \ldots, n\}$, the shortened code $C_L^{\perp e}$ of $C^{\perp e}$ is $\{\vec{x} \in C^{\perp e}$: components of $\vec{x}$ are zero outside $L\}$.

The i-th component has an erasure. $J \subset \{1, \ldots, n\}$, where $i \in J$, is a recovery set of size $r + \delta - 1$. Erasure indices belong to $I \subset J$. There are t errors and e erasures in J with $2t + e < \delta$.

1. Consider components in $J \setminus I$, which may have errors. Correct those errors in $J \setminus I$ by measuring observables defined by $C_{J \setminus I}^{\perp e}$, which identify errors in $J \setminus I$, followed by an error-correcting operation acting on $J \setminus I$.

Localized error and erasure correction (quantum case) II

2. Now components in $J \setminus I$ are error-free.
3. Erased components in I are reconstructed from those in $J \setminus I$.

- The above procedure uses components only in J.

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