New Discretized Schemes of 1st Mur's ABC and Its Applications to Scattering Problems

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Abstract- The new discretized schemes of Mur's absorbing boundary condition (ABC) based on finite difference frequency domain method has been presented in this paper. The analysis figures out that their numerical accuracy are second-order. Combining the new schemes with five-point format of Helmholtz equation to solve scattering problems, the numerical results show that the accuracy of numerical solution with the new schemes is higher than that with ordinary discretized scheme of Mur's absorbing boundary condition.

I. INTRODUCTION

Finite Difference Method (FD) is one of the most prevailing methods on computational electromagnetics due to its simplicity and low cost. In this paper finite difference frequency domain (FDFD) is used to analyze cylinder scattering problem. Generally, the scattering region is infinite, so it will be truncated. At the truncation boundary, 1st Mur's absorbing boundary condition (ABC) will be set. The ABC is a Robin's condition. Its numerical discretized equations are of first-order accuracy. For Helmholtz equation, its numerical accuracy of discretized equations is second-order. So the numerical accuracies of Helmholtz equation and ABC condition are mismatched. In order to improve the numerical accuracy of ABC, the new numerical discretized equations of ABC are presented here, and they have second-order numerical accuracy. Then, two kinds of numerical discretized equations of ABC with different numerical accuracies are employed to solve the scattering problems. The numerical results show that the numerical solution with second-order numerical precision equations of ABC is better.

II. NEW NUMERICAL SCHEME OF MUR BOUNDARY CONDITION

Mur boundary condition, which proposed by Gerrit Mur [1] in 1981 based on outgoing wave theory [3], is very easy to implement and has quite good results if properly applied, so until nowadays this type of boundary conditions still widely used in both FDTD and FDFD. Due to including first-order derivative, its discretized numerical equations are only first-order accuracy. In order to enhance its numerical accuracy, some researches use second-order Mur condition. However, it is not only very complicate, but also it is still first-order numerical accuracy. In fact, boundary condition including second-order derivative is not reasonable for Helmholtz equation. Here, the new discretized numerical equations for Mur's ABC will be presented and they are of second-order numerical accuracy.

Considering the scattering problem of a PEC square cylinder with infinite length and with TM_z plane wave incidence, it can be expressed as follow 2-D problem

$$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}+k^2\phi=0,(x,y)\in\Omega & (1)\\ \phi=0, \text{on } \Gamma_1 & (2)\\ \dfrac{\partial\phi}{\partial n}+jk\phi=0, on\, \Gamma_2 & (3)\end{cases}$$

Where ϕ=E_z here. As Fig. 1 shows, Ω is the cross section of scattering area, Γ_1 is the PEC boundary, Γ_2 is the truncated boundary, n is outgoing unit normal vector of Γ_2 . Equation (3) is the well-known 1st Mur's absorbing boundary condition.

Just for simplicity, only the left boundary (x=0) of Γ_2 will be considered here. Equation (3) is usually discretized as:

$$\frac{\partial\phi_0}{\partial x}-jk\phi_0=\frac{\phi_2-\phi_0}{h}-jk\phi_0+O(h) \qquad (4)$$

Here, $\phi_0=\phi(x_0,y_0)$ is the left boundary point shown in Fig.2, and $\phi_2=\phi(x_0+h,y_0)$, h is mesh step. Apparently, equation (4) has only first-order numerical accuracy，while the traditional 5 nodes format of Helmholtz equation has second-order numerical accuracy, their precisions are mismatched.

Considering the condition that quantity ϕ satisfies equation (1) at all points in computational domain, so do the points at boundary (not include the corner points). As Fig. 2 (a) shows, use Taylor expansion of point 0 (x_0,y_0) at 1, 2, 3:

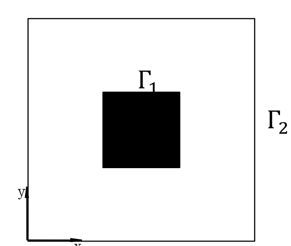
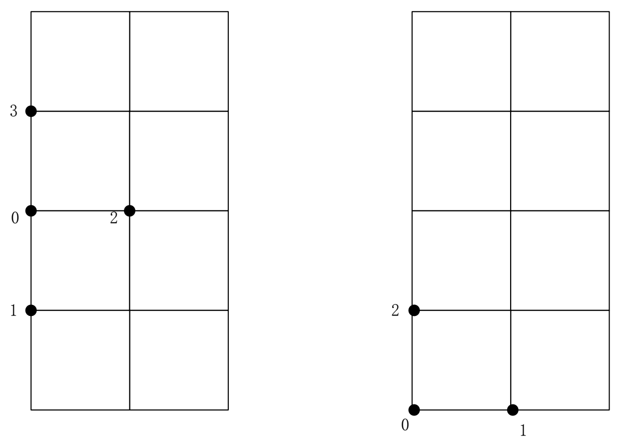


Figure 1. The cross sections of the scattering area for example 2



(a) (b)

Figure 2. Mesh and nodes for 1st Mur ABC

$$\phi_1 = \phi(x_0, y_0) - \frac{\partial \phi(x_0, y_0)}{\partial y} h + \frac{1}{2!} \frac{\partial^2 \phi(x_0, y_0)}{\partial y^2} h^2 + O(h^3) \tag{5}$$

$$\phi_2 = \phi(x_0, y_0) + \frac{\partial \phi(x_0, y_0)}{\partial x} h + \frac{1}{2!} \frac{\partial^2 \phi(x_0, y_0)}{\partial x^2} h^2 + O(h^3) \tag{6}$$

$$\phi_3 = \phi(x_0, y_0) + \frac{\partial \phi(x_0, y_0)}{\partial y} h + \frac{1}{2!} \frac{\partial^2 \phi(x_0, y_0)}{\partial y^2} h^2 + O(h^3) \tag{7}$$

Such that:

$$\phi_1 + \phi_3 + 2\phi_2 - 4\phi_0 = \frac{\partial^2 \phi(x_0, y_0)}{\partial y^2} h^2 + \frac{\partial^2 \phi(x_0, y_0)}{\partial x^2} h^2 + 2 \frac{\partial \phi(x_0, y_0)}{\partial x} h + O(h^3) \tag{8}$$

ϕ also satisfies Helmholtz equation at (x_0, y_0), so

$$\frac{\partial^2 \phi(x_0, y_0)}{\partial y^2} h^2 + \frac{\partial^2 \phi(x_0, y_0)}{\partial x^2} h^2 = -k^2 h^2 \phi(x_0, y_0) \tag{9}$$

Substitute formula (9) into formula (8), obtain the formula below:

$$\phi_1 + \phi_3 + 2\phi_2 - 4\phi_0 + k^2 h^2 \phi_0 - 2 \frac{\partial \phi(x_0, y_0)}{\partial x} h = O(h^3) \tag{10}$$

Therefore

$$\frac{\partial \phi(x_0, y_0)}{\partial x} = \frac{1}{2h} (\phi_1 + \phi_3 + 2\phi_2 - 4\phi_0 + k^2 h^2 \phi_0) + O(h^2) \tag{11}$$

Substitute formula (11) into formula (3) (noticed n=-x), obtain:

$$\frac{\partial \phi(x_0, y_0)}{\partial x} - jk\phi(x_0, y_0) = \frac{1}{2h} (\phi_1 + \phi_3 + 2\phi_2 - 4\phi_0 + k^2 h^2 \phi_0) - jk\phi(x_0, y_0) + O(h^2) \tag{12}$$

So the new scheme of 1st Mur left boundary points except corner points is：

$$\phi_1 + \phi_3 + 2\phi_2 - 4\phi_0 + k^2 h^2 \phi_0 - 2jkh\phi_0 = 0 \tag{13}$$

Comparing to formula (4), the formula (13) has second-order numerical accuracy obviously. So if using this scheme to discrete 1st Mur boundary, it can obtain better accuracy than that of formula (4) theoretically. Same procedure can be used to derivate the scheme of 1st Mur boundary condition at other nodes of the boundary, and the same result like (13) can be obtained.

At the corner point of computation domain in Fig.2 (b), the above procedure can be used similarly. Considering that the corner point has two different normal direction, a weighted absorbing boundary condition, which combine 1st Mur boundary at both normal direction is constructed here:

$$\alpha \left(\frac{\partial \phi_0}{\partial x} - jkE_{z_0} \right) + (1 - \alpha) \left(\frac{\partial \phi_0}{\partial y} - jk\phi_0 \right) = 0 \tag{14}$$

$$\phi_1 = \phi(x_0, y_0) + \frac{\partial \phi(x_0, y_0)}{\partial x} h + \frac{1}{2!} \frac{\partial^2 \phi(x_0, y_0)}{\partial x^2} h^2 + O(h^3) \tag{15}$$

$$\frac{\partial \phi(x_0, y_0)}{\partial x} = \frac{1}{h} \left(\phi_1 - \phi(x_0, y_0) - \frac{1}{2!} \frac{\partial^2 \phi(x_0, y_0)}{\partial x^2} h^2\right) + O(h^2) \tag{16}$$

$$\phi_2 = \phi(x_0, y_0) + \frac{\partial \phi(x_0, y_0)}{\partial y} h + \frac{1}{2!} \frac{\partial^2 \phi(x_0, y_0)}{\partial y^2} h^2 + O(h^3) \tag{17}$$

$$\frac{\partial \phi(x_0, y_0)}{\partial y} = \frac{1}{h} \left(\phi_2 - \phi(x_0, y_0) - \frac{1}{2!} \frac{\partial^2 \phi(x_0, y_0)}{\partial y^2} h^2\right) + O(h^2) \tag{18}$$

Substitute formula (16), (18) into formula (14), gets:

$$
\alpha\left(\frac{\partial\phi_0}{\partial x}-jk\phi_0\right)+(1-\alpha)\left(\frac{\partial\phi_0}{\partial y}-jk\phi_0\right)
=\frac{\alpha}{h}\left(\phi_2-\ \phi_0-\frac{1}{2!}\frac{\partial^2\phi_0}{\partial x^2}h^2-jkh\phi_0\right)
+\frac{1-\alpha}{h}\left(\phi_1-\ \phi_0-\frac{1}{2!}\frac{\partial^2\phi_0}{\partial y^2}h^2-jkh\phi_0\right)
+O(h^2) \quad (19)
$$

In order to eliminate the second order terms by using (2), select $\alpha = 0.5$, then formula (19) becomes:

$$
\alpha\left(\frac{\partial\phi_0}{\partial x}-jk\phi_0\right)+(1-\alpha)\left(\frac{\partial\phi_0}{\partial y}-jk\phi_0\right)
=\frac{1}{2h}(\phi_1+\phi_2-2\phi_0+0.5h^2k\phi_0
-2jkh\phi_0)
+O(h^2) \quad (20)
$$

So the new scheme at this corner points is:

$$
\phi_1+\phi_2-2\phi_0+0.5k^2h^2\phi_0-2jk\phi_0h=0 \quad (21)
$$

This formula holds at both four corner points.

The formula (13) and (21) are the new discretized scheme of 1st Mur's absorbing boundary condition. Comparing to formula (4), they have second-order numerical accuracy.

III. Numerical Results

A. Example 1: square cylinder scattering problem

A scattering problem of PEC square cylinder with infinite length (Fig. 1) which discussed above is computed here to verify the efficiency of the new scheme of Mur boundary condition derived above. Let λ be the wavelength and the mesh step h is $1/30\lambda$, the side length of square cylinder is $2/5\ \lambda$. The width of square domain along *x*-axis is $6/5\ \lambda$. Considering TM_z plane wave incidence along the *x*-axis, so only z direction electric field existed, and this problem can be expressed just like equations (1), (2), (3). At the truncated boundary, the traditional 1st Mur boundary discretized by formula (4) and (13), (21) respectively, noticed that when using (4), the corner is handled with mean absorption field conditions [6]. The numerical result of absolute value of total field with the new scheme of 1st Mur ABC is shown in Fig. 3.

In order to verify the effect of the new scheme, a finer mesh with step $h_F = 0.5h$ is used to calculate the same problem with the traditional 1stMur boundary condition, and its result is considered as reference solution. Comparing with the reference solution, the relative errors of numerical solutions computed by tradition scheme (formula (4)) and new scheme (formula (13) and (21)) of 1st Mur ABC respectively are presented in Fig.4.

Table I lists the mean relative error, maximum relative error, mean square error and iteration numbers of both scheme.

According to numerical results of Figure 4 and table I, formula (13) and (21), the new scheme of discretized equations for Mur ABC, can obtained more accurate numerical solution.

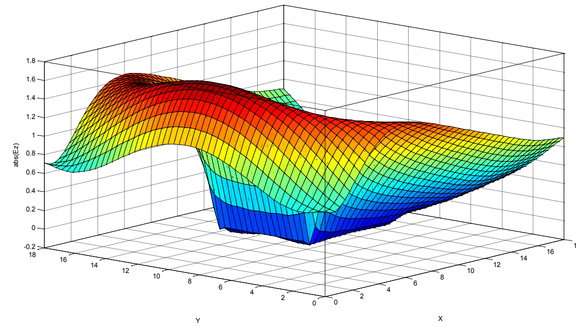
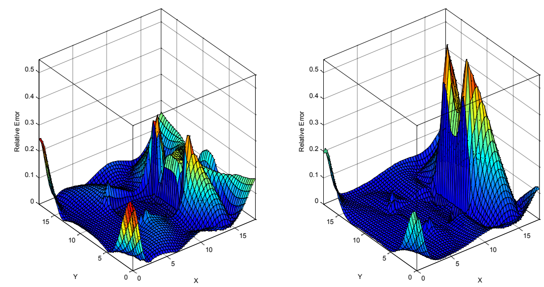


Figure 3 Numerical solution of total field for example 1



(a) (b)

Figure 4 Relative errors comparing to the reference solution. (a) New scheme of 1st Mur ABC. (b) Traditional 1st Mur ABC

TABLE I

RESULT OF EXMAPLE 1

	Mean Relative Error	Maximum Relative Error	Mean Square Error	Iteration Numbers
Traditional Mur	0.0576	0.5506	0.1175	1106
New Scheme	0.0480	0.3093	0.0749	965

B. Example 2: Two circle cylinders scattering problem

Another scattering problem of two PEC circle cylinders with infinite length is analyzed here. Fig. 7 is the cross section of the scattering area contains the cylinders. Such as the example I above, let mesh step $h = 1/30\,\lambda$, and the diameter of two cylinders is $10h$. The distance between both cylinders is $18h$. Two schemes of 1st Mur's ABC, i.e. formula (4), (13) and (21) are used to solve the scattering problem. The reference solution is solved with finer mesh $h_F = 1/60\lambda$. Here, the Compatible Sub-gridding Method [6] is used to approximate the circular boundary. The total field numerical solutions shown in Fig. 5, Comparing with the reference solution, the relative errors are shown in Fig.6. The mean relative errors, the maximum errors, the mean square errors and the iteration numbers for both scheme are presented in Table II.

The numerical results of Figure 5 and table II also validate that using the formula (13), (21) at the boundary can obtained more accurate numerical solution than traditional scheme.

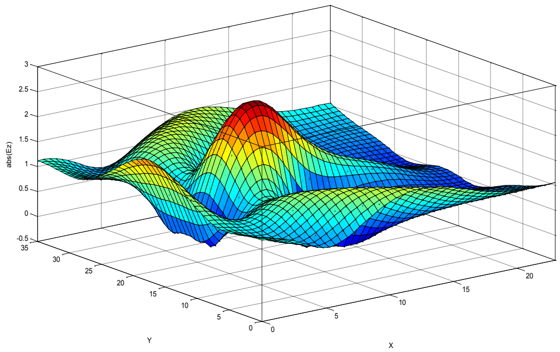
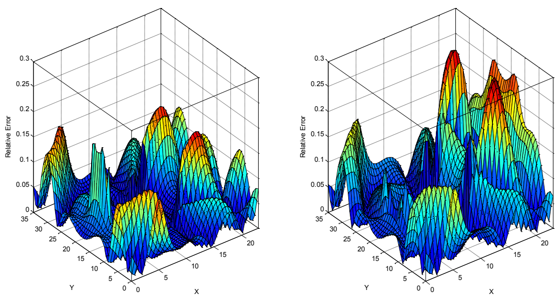


Figure 5. The numerical solution of total field for example 2



(a) (b)

Figure 6. .Relative errors comparing to the reference solution. (a) New scheme of 1st Mur. (b) Traditional 1st Mur

TABLE II

RESULT OF EXMAPLE 2

	Mean Relative Error	Maximum Relative Error	Mean Square Error	Iteration Numbers
Traditional Mur	0.0675	0.2838	0.0867	23697
New Scheme	0.0440	0.2012	0.0571	21663

IV. Conclusion

In order to improve the accuracy of numerical solution for scattering problem, the new discretized equations of Mur's ABC, i.e. formula (13) and (21), are presented in this paper. Theoretically, the new discretized equations are of second-order numerical accuracy. Combining the new equations with the normal 5 points format of Helmholtz equation, two numerical examples of scattering problems are solved. Comparing to traditional discretized equations of Mur's ABC which is first-order numerical accuracy, the results show that the new scheme can significantly improve the accuracy of numerical solution.

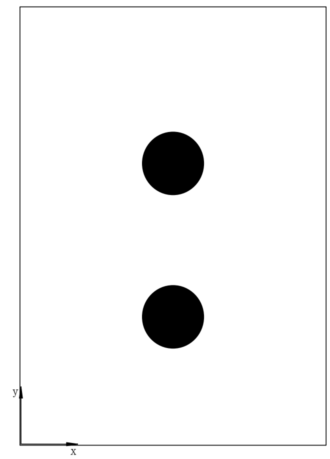


Figure 7. The cross sections of the scattering area for example 2

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