Error Analysis of a Novel Absorbing Boundary Condition for the 3-Step LOD-FDTD Method

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***Abstract*-In this paper, the estimation of the numerical error of a novel absorbing boundary condition (ABC) proposed for the split-step finite-difference time-domain (FDTD) method and the incident angle of the plane wave is investigated. Compared with two other previously reported Mur first-order ABCs, the proposed ABC is simpler and requires less additional computation. Besides, it also has much better absorption performance.**

I. Introduction

The finite-difference time-domain (FDTD) method is widely used for solving electromagnetic-wave problems [1]. The split-step (SS) FDTD method and the locally one-dimensional (LOD) FDTD [2]-[4] have attracted much attention recently thanks to their unconditional stability, simplicity and flexibility. In [2], the 3-D LOD-FDTD is split into a two-step approach but is complicated in its formulations. And recently it is split into a three-step approach with reduced computational complexity [3]. In [4], the LOD-FDTD method has the second-order temporal accuracy and involves simpler updating procedures.

When the LOD-FDTD methods are applied to open structures, truncated boundary conditions are required to convert the unlimited physical space to the limited calculating space. Generally, there are two kinds of truncated boundary conditions. One is the employment of nonphysical absorbing media, such as the perfect matched layer (PML) [5]-[7]. The other one is the ABC [8]-[12] based on travelling wave equations. Compared with the PMLs, the ABCs require much less computation and memory at the cost of introducing greater reflection.

In [10], a consistent implementation of Mur ABC into three-step 3-D LOD-FDTD method is introduced, which exhibits better wave-absorbing capability than the conventional schemes. In [11], a novel ABC is proposed according to the intrinsic updating procedures in the SS-FDTD method. The proposed ABC requires less additional computation and causes less reflection than the traditional Mur first-order ABCs.

In this paper, the estimation of the numerical error of the novel ABC [11] for the 3-Step LOD-FDTD method is analyzed. The proposed ABC is simpler and requires less additional computation. In addition, it has much better absorption performance compared with two other Mur first-order ABCs.

II. Formulations

Maxwell's equations in isotropic and lossless media are given as

$$\nabla\times H=\varepsilon\frac{\partial E}{\partial t}\quad \nabla\times E=-\mu\frac{\partial H}{\partial t}. \tag{1}$$

For simplicity, only the 2D TM(y) waves are considered. Therefore, these equations can be expressed in the Cartesian coordinates as

$$\frac{\partial U}{\partial t}=\mathbf{A}U+\mathbf{B}U \tag{2}$$

where

$$\mathbf{A}=\begin{bmatrix}0 & \frac{1}{\varepsilon}\frac{\partial}{\partial z} & 0\\ \frac{1}{\mu}\frac{\partial}{\partial z} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\quad \mathbf{B}=\begin{bmatrix}0 & 0 & -\frac{1}{\varepsilon}\frac{\partial}{\partial x}\\ 0 & 0 & 0\\ -\frac{1}{\mu}\frac{\partial}{\partial x} & 0 & 0\end{bmatrix}$$

and $\left[U\right]=\left[E_y,H_x,H_z\right]^T$.

It should be noted that **A** and **B** are sparse matrices whose elements are related to the spatial derivatives along the x- and z-directions, respectively. The updating procedure of the three-step LOD-FDTD can be written as follows [2]:

$$\left(\mathbf{I}-\frac{\Delta t}{4}\mathbf{A}\right)U^{n+\frac{1}{4}}=\left(\mathbf{I}+\frac{\Delta t}{4}\mathbf{A}\right)U^{n} \tag{3a}$$

$$\left(\mathbf{I}-\frac{\Delta t}{2}\mathbf{B}\right)U^{n+\frac{3}{4}}=\left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{B}\right)U^{n+\frac{1}{4}} \tag{3b}$$

$$\left(\mathbf{I}-\frac{\Delta t}{4}\mathbf{A}\right)U^{n+1}=\left(\mathbf{I}+\frac{\Delta t}{4}\mathbf{A}\right)U^{n+\frac{3}{4}} \tag{3c}$$

According to [2] and [10], (3) can be rearranged as

$$\frac{\partial}{\partial t}U^{n+\frac{1}{8}}=2\mathbf{A}U^{n+\frac{1}{8}} \tag{4a}$$

$$\frac{\partial}{\partial t}U^{n+\frac{1}{2}}=2\mathbf{B}U^{n+\frac{1}{2}} \tag{4b}$$

$$\frac{\partial}{\partial t}U^{n+\frac{7}{8}}=2\mathbf{A}U^{n+\frac{7}{8}} \tag{4c}$$

For conciseness, the first updating step of the LOD-FDTD method is presented in detail. The second and third updating step can be derived in the same way.

More specifically, substituting field components into (3a) reads

$$E_y^{n+\frac{1}{4}} = E_y^n + \frac{\Delta t}{4\varepsilon}\left[\frac{\partial H_x^{n+\frac{1}{4}}}{\partial z} + \frac{\partial H_x^n}{\partial z}\right] \tag{5a}$$

$$H_x^{n+\frac{1}{4}} = H_x^n + \frac{\Delta t}{4\mu}\left[\frac{\partial E_y^{n+\frac{1}{4}}}{\partial z} + \frac{\partial E_y^n}{\partial z}\right] \tag{5b}$$

$$H_z^{n+\frac{1}{4}} = H_z^n \tag{5c}$$

This scheme at the z=1 grid is illustrated. To simplify the calculating process, some approximations are made.

$$E_y^{n+\frac{1}{4}}(i,1) - E_y^n(i,1) \approx \frac{\Delta t}{2\varepsilon\Delta z}[H_x^n(i,1) - H_x^n(i,0)] \tag{6a}$$

$$H_x^{n+\frac{1}{4}}(i,1) - H_x^n(i,1) \approx \frac{\Delta t}{2\mu\Delta z}[E_y^{n+\frac{1}{4}}(i,1) - E_y^{n+\frac{1}{4}}(i,0)] \tag{6b}$$

$$H_z^{n+\frac{1}{4}}(i,1) = H_z^n(i,1) \tag{6c}$$

Note that in the above formulations (6a)-(6c), all equations are explicit, therefore reducing computational complexity.

In the subsections, the ABC implementations for E_y variable at the boundary z=0 are devised. The field components at other boundaries can be derived similarly.

A. The proposed ABC

In the first updating step, for an outgoing wave at boundary z=0, according to equation (4a) one can obtain that

$$\frac{\partial E_y^{n+\frac{1}{8}}}{\partial t} = -2c\frac{\partial E_y^{n+\frac{1}{8}}}{\partial z} \tag{7}$$

where $c = 1/\sqrt{\mu\varepsilon}$ is the wave velocity.

Equation (7) can be discretized as

$$E_y^{n+\frac{1}{4}}(i,0) + \xi_1 E_y^{n+\frac{1}{4}}(i,1) = \xi_1 E_y^n(i,0) + E_y^n(i,1) \tag{8}$$

where $\xi_1 = (2\Delta z - c\Delta t)/(2\Delta z + c\Delta t)$.

In the second updating step, according to the method in [11], at the boundary z=0, E_y does not need to be updated.

$$E_y^{n+\frac{3}{4}}(i,0) = E_y^{n+\frac{1}{4}}(i,0) \tag{9}$$

The third updating step is similar to the first updating step.

$$E_y^{n+1}(i,0) + \xi_1 E_y^{n+1}(i,1) = \xi_1 E_y^{n+\frac{3}{4}}(i,0) + E_y^{n+\frac{3}{4}}(i,1) \tag{10}$$

B. The ABC in [10]

Follow the method in [10], one can obtain the ABC in each updating step as

$$E_y^{n+\frac{1}{4}}(i,0) + \xi_1 E_y^{n+\frac{1}{4}}(i,1) = \xi_1 E_y^n(i,0) + E_y^n(i,1) \tag{11}$$

$$E_y^{n+\frac{3}{4}}(i,0) + E_y^{n+\frac{3}{4}}(i,1) = E_y^{n+\frac{1}{4}}(i,0) + E_y^{n+\frac{1}{4}}(i,1) \tag{12}$$

$$E_y^{n+1}(i,0) + \xi_1 E_y^{n+1}(i,1) = \xi_1 E_y^{n+\frac{3}{4}}(i,0) + E_y^{n+\frac{3}{4}}(i,1) \tag{13}$$

C. Conventional Mur ABC

At boundary z=0, a one-way outgoing wave equation is as follow:

$$\frac{\partial E_y^{n+\frac{1}{8}}}{\partial t} = -c\frac{\partial E_y^{n+\frac{1}{8}}}{\partial z} \tag{14}$$

Discretization of (14) leads to

$$E_y^{n+\frac{1}{4}}(i,0) + aE_y^{n+\frac{1}{4}}(i,1) = aE_y^n(i,0) + E_y^n(i,1) \tag{15}$$

Similarly, in the second and third updating step one can obtain

$$E_y^{n+\frac{3}{4}}(i,0) + bE_y^{n+\frac{3}{4}}(i,1) = bE_y^{n+\frac{1}{4}}(i,0) + E_y^{n+\frac{1}{4}}(i,1) \tag{16}$$

$$E_y^{n+1}(i,0) + aE_y^{n+1}(i,1) = aE_y^{n+\frac{3}{4}}(i,0) + E_y^{n+\frac{3}{4}}(i,1) \tag{17}$$

where $a = (4\Delta z - c\Delta t)/(4\Delta z + c\Delta t)$

$b = (2\Delta x - c\Delta t)/(2\Delta x + c\Delta t)$.

III. NUMERICAL RESULTS

The numerical error of the ABC, which includes the error introduced by the approximation of the equation and the error introduced by truncation of the expression of the field, can be derived by virtue of a homochromous plane wave as:

$$E_y^n(i,k) = E_0 e^{j(k_x i\Delta x + k_z \Delta z + jwn\Delta t)}$$

All propagation directions are considered to study the numerical error of the ABC. Let k_x=$k$cosθ, k_z=$k$sinθ and θ is incident angle with respective to x-axis, as shown in Fig. 1.

A 2-D computation domain is studied and the cell size in each direction is 1mm. Note that CFLN is defined as the ratio between the time step in the SS-FDTD and the maximum CFL limit in the standard FDTD. The numerical error is calculated as $\left|E_{observed} - E_{ref}\right|/\max(\left|E_{ref}\right|)$, where E_{ref} is obtained at the same position with an extended time step that virtually generates no reflection.

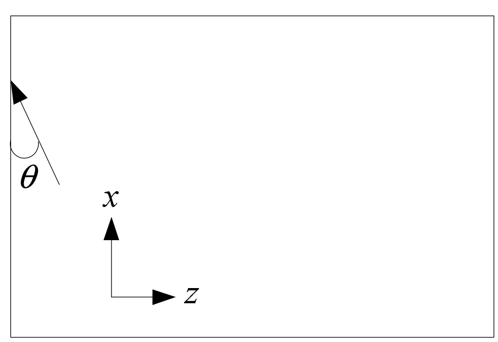

Figure 1. Wave propagates upon the ABC at z=0

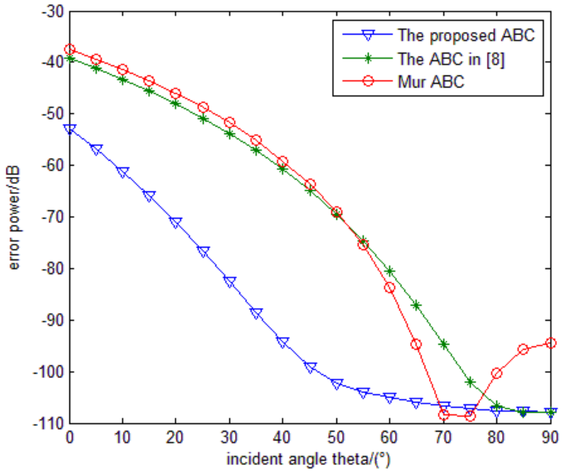

Figure 2. Numerical errors of different ABCs with different incident angles.

The effects of wave propagation direction on the numerical error of the proposed ABC, the ABC in [10] and the conventional Mur ABC are investigated.

Fig. 2 shows the relationship between numerical errors and the incident angle of the plane wave under the conditions f=10GHz and CFLN=1.

According to Fig. 2, the proposed ABC has the best absorption performance compared with the ABC in [10] and the conventional Mur ABC.

IV. Conclusions

The estimation of the numerical error of a novel ABC [11] proposed for the split-step finite-difference time-domain (FDTD) method and the incident angle of the plane wave is investigated. The proposed ABC is simpler and has much better absorption performance than two other traditional Mur first order ABCs. Therefore, the proposed ABC can be used conveniently in the split-step FDTD method to solve open structure EM problems.

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