Multi-Population Replicator Dynamics with Changes of Interpretations of Strategies

Takafumi KANAZAWA† and Toshimitsu USHIO†

†Graduate School of Engineering Science, Osaka University
1-3 Machikaneyama, Toyonaka, Osaka 560-8531 Japan,
Email: kanazawa@hopf.sys.es.osaka-u.ac.jp, ushio@sys.es.osaka-u.ac.jp

Abstract—In this paper, we propose multi-population replicator dynamics with changes of players’ interpretations. We consider that each population has various interpretation functions and choose one of them depending on payoffs. We propose a model representing changes of the interpretation functions according to the payoffs. Moreover, we apply our proposed model to a well-known example of a hypergame “Soccer Hooliganism” and show that the interpretation changes depending on the evolutions of distributions of strategies.

1. Introduction

The game theory has made considerable achievements in many scientific fields including economics, politics, and social science. Its extensions have been investigated in several different directions. Among them, we focus on an evolutionary game and a hypergame [1, 2].

In evolutionary game theory, the distribution of strategies in the population is changed according to payoffs which players earn depending on their selected strategies [3]. In this process, a strategy is called an evolutionarily stable strategy (ESS) if, whenever all members of the population adopt it, no dissident strategy could invade the population [4]. On the other hands, a dynamic characteristics of the selection process is described by replicator dynamics [5, 6, 7]. Replicator dynamics describes evolutions of the distribution of strategies in the population itself. Though no players’ rationalities are assumed in evolutionary games, an ESS and an equilibrium point of replicator dynamics are closely involved with equilibrium concepts of noncooperative game theory [8].

On the other hand, in many conflict situations, players’ perceptions are different each other and their behaviors depend on their perceptions to the game. Such a situation is modeled by a hypergame [2, 9, 10]. Various solution concepts and methods of analysis taking players’ perceptions into account have been proposed for the hypergames [11]. The hypergame theory investigates the effects of players’ perceptions about strategies and payoffs on their decisions.

Replicator dynamics describes evolutions of the distribution of strategies in the populations according to payoffs which players earn depending on their selected strategies. Although replicator dynamics is a model of players that select a strategy by trial and error, the influence of players’ perceptions has to be taken into consideration. Recently, to discuss such a situation, replicator dynamics for multi-population hypergames has been proposed in [12]. A rigorous definition of player’s perception is given by an interpretation function which specifies the relationship between strategies of a player and those of the other players. Replicator dynamics for hypergames is a model introducing the interpretation function into multi-population evolutionary games. In n-population models where each population has different perception, both an ESS and replicator dynamics are formulated [12]. However, it is assumed that players’ perceptions are given and fixed. Each player’s perception may change according to strategies used by the other players and results of games.

In this paper, we propose multi-population replicator dynamics with changes of players’ interpretations. We consider that each population has various interpretation functions and choose one of them depending on payoffs. We propose a model representing changes of the interpretation functions. Soccer Hooliganism [10] is a well-known example of a hypergame, and its replicator dynamics has been derived in [12]. We apply our proposed model to the example and show that the interpretation changes depending on the distributions of strategies.

2. Replicator Dynamics for Hypergames

A common perception in a population may not always acceptable in other populations. For example, if some differences of perceptions arise between populations, then strategies which are regarded as the same strategy in a population may be perceived distinguishably in the other populations. In order to discuss such situations, replicator dynamics for multi-population hypergames has been proposed [12].

Suppose that there exist n large populations $P_1, P_2, \ldots, P_n$. For each population $P_i (i \in N = \{1, \cdots, n\})$, let $\Phi_i = \{1, 2, \ldots, m_i\}$ be a set of pure strategies of $P_i$, and $S_i$ be a set of population states of $P_i$, where a population state is a distribution of strategies in a population of players with a pure strategy. In our proposed model, the players’ interpretations of the set of pure strategies and population states can be different from one another. Let
Φij and Si,j be Pi’s sets of pure strategies and population states perceived by Pi, respectively. The first subscript i stands for a population to which the strategy (or the population state) belongs, while the second one j stands for a population which perceives the strategy. Denoted by

\[ s_i = (s_i^1, \ldots, s_i^n) \in S_i \] is a population state of Pi, where \( s_i^k \) is the proportion of players with a pure strategy \( k \in \Phi_i \) in Pi. We define sets of population state combinations \( S \) and \( S_i \) by Cartesian product \( S = \times_{i \in N} S_i \) and \( S_i = \times_{j \in N \setminus i} S_j \), respectively, where \( (N - i) = \{ j \in N | j \neq i \} \). For \( s \in S \), denoted by \( s_i \in S_i \), is the population state combination which results from \( s \) by removing \( s_i \in S_i \): i.e., \( s_i = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \). Let \( R_i : \times_{j \in N} S_j \rightarrow R \) be the payoff function for population \( P_i \), where \( S_i = S \).

The mapping \( F_{ij} : \Phi_i \rightarrow \Phi_j \) is called \( F_{ij} \)’s interpretation function to \( P_i \). Depending on the interpretation function \( F_{ij} \) of pure strategies, an interpretation function \( f_{ij} : S_i \rightarrow S_j \) of population states is defined as follows: for each population state \( s_i \in S_i ,

\[ f_{ij} = \sum_{h \in \Phi_i | F_{ij}(h) = k} s^h_i, \quad k \in \Phi_j, \quad (1) \]

where \( f_{ij}^k \) is the kth element of \( f_{ij} \) and \( f_{ii} \) is the identity mapping \( f_{ii}(s) = s \). We define \( F_{ij} \)’s interpretation of a population state combination \( s \) by \( f_{ij}(s) = (f_{ij}^1(s_1), \ldots, f_{ij}^n(s_n)) \).

In each population, suppose that the rate of increase of players with a strategy \( k \) is expressed as the difference between the payoffs of a player with a strategy \( k \) and the average payoff of the population. Replicator dynamics for hypergames which describes the changes of the population states in each population is given by

\[ s^k_i = s^k_i \left[ R_i(f_i(e^k_i, \ldots)), R_i(f_i(s)) \right] \]

for all \( i \in N \) and \( k \in \Phi_i \), where \( e^k_i \) is the m-dimensional unit vector such that the kth element equals 1.

### 3. Changes of Interpretations

The model of Section 2 is an evolutionary game model considering players’ perceptions [12]. However, it is assumed that the introduced interpretation functions are given and fixed. In this section, we propose a model with changes of the interpretation functions.

We consider that each population has various interpretation functions and changes its interpretations depending on payoffs. Let \( f(s) = (f_i(s), f_2(s), \ldots, f_n(s)) \) be an interpretation function combination, where \( f_j \in \{ f_1^1, f_1^2, \ldots, f_m^j \} \) is an element of set of \( P_i \)’s interpretation of \( i \). Denoted by \( \sigma_{i}^{jk} \) is the event that \( P_i \) changes its interpretation function from \( f^j \) to \( f^k \). Let \( \Sigma \) be a set of all such events. We assume that the event \( \sigma_{i}^{jk} \in \Sigma \) occurs when \( P_i \) uses \( f^j \) and the difference between the average payoff of \( P_i \) with \( f^j \) and that with \( f^k \) becomes equal or greater than \( \lambda_{i}^{jk} > 0 \). Thus, \( P_j \) changes its interpretation function from \( f^j \) to \( f^k \) if there exist \( k \) and \( k' (k, k' = 1, \ldots, m_i) \) satisfying

\[ \sigma_{i}^{jk} \in \Sigma \quad \text{and} \quad R_i \left( f^k(s) \right) - R_i \left( f^j(s) \right) \geq \lambda_{i}^{jk}. \]

Note that \( P_i \) must change its interpretation functions if the condition (3) holds. For a population state combination \( s \) and an interpretation combination \( f \), if there exist \( i (i \in N), k, \) and \( k' (k, k' = 1, \ldots, m_i) \) satisfying the condition (3), \( s \) is said to be infeasible for \( f \), otherwise feasible for \( f \).

According to the above definition, interpretation functions of each population change discretely, and evolutions of population states depend on replicator dynamics (2). Obviously, such a system is a hybrid system.

### 4. Soccer Hooliganism

Soccer Hooliganism [10], which is a noncooperative game between fans and authorities, is a well-known example of a hypergame and its replicator dynamics has been derived in [12]. In this section, we introduce changes of interpretations into the replicator dynamics.

In the game perceived by fans, which will be called “fans’ game”, the fans have three strategies, “Ordinary behavior (Beh.)”, “Play hooligan (P.H.)”, and “Real hooliganism (R.H.)”. Authorities have two strategies, “Tough response (Tou.)” and “Tolerant response (Tol.)”. On the other hand, in the game perceived by the authorities, which will be called “authorities’ game”, the fans have two strategies, “Acceptable (Acc.)” and “Unacceptable (Unacc.)”, and the authorities have two strategies, “Intervention (Int.)” and “Non-Intervention (Non-Int.)”.

In fans’ game, population states are described as follows:

- Fans’ population state \( s_1 = [x_1^1, x_1^2, x_1^3] \):
  - \( x_1^1 \) : a proportion of players with the strategy “Beh.”,
  - \( x_1^2 \) : a proportion of players with the strategy “P.H.”,
  - \( x_1^3 \) : a proportion of players with the strategy “R.H.”.

- Authorities’ population state \( s_2 = [x_2^1, x_2^2]^T \):
  - \( x_2^1 \) : a proportion of players with the strategy “Tou.”,
  - \( x_2^2 \) : a proportion of players with the strategy “Tol.”.

In authorities’ game, population states are described as follows:

- Fans’ population state \( s_{12} = [s_{12}^1, s_{12}^2] \):
  - \( s_{12}^1 \) : a proportion of players with the strategy “Acc.”,
  - \( s_{12}^2 \) : a proportion of players with the strategy “Unacc.”.

- Authorities’ population state \( s_2 = [s_1^1, s_1^2]^T \):
  - \( s_1^1 \) : a proportion of players with the strategy “Int.”,
  - \( s_1^2 \) : a proportion of players with the strategy “Non-Int.”.

We consider two types of interpretation function combinations. In both cases, suppose that fans perceive “Int.” and “Non-Int.” as “Tou.” and “Tol.”, respectively. Hence, \( f(s) = (s_1, s_2) \). Moreover, in the type 1, suppose that authorities perceive both “Beh.” and “P.H.” as “Acc.”, and
only “R.H.” as “Unacc.”. Hence, \( f^1_2(s) = ([s^1_1 + s^3_1, s^1_2]^T, s_2) \).
In the type 2, however, suppose that authorities only perceive “Beh.” as “Acc.”, and perceive the rest as “Unacc.”. Hence, \( f^2_2(s) = ([s^1_1, s^3_1 + s^3_2]^T, s_2) \).

**Simulation I** Perceived payoffs on Soccer Hooliganism is shown in Table 1, where each cell indicates perceived payoffs, and the 1st and the 2nd number in each cell are payoffs of fans and authorities, respectively.

Note that \( \Sigma = s^1_1 + s^2_1 = 1 \). From Eq. (2), replicator dynamics of \( P_1 \)'s strategies are formulated as follows:

\[
\begin{align*}
\dot{s}_1^1 &= s_1^1 \left[ (2 + s_2^1) (s_1^1 - 1) - (1 - 3 s_2^1) s_1^1 \right], \quad (4) \\
\dot{s}_1^2 &= s_1^2 \left[ (2 + s_2^2) s_1^1 + (1 - 3 s_2^2) (1 - s_1^1) \right]. \quad (5)
\end{align*}
\]

For each interpretation function of \( P_2 \), dynamics of \( P_2 \)'s strategies are given by

\[
\dot{s}_2^1 = \begin{cases} 
{s_2^1(1 - s_1^1)(1 - 2 s_1^1 - 2 s_2^1) & \text{for } f^1_2, \\
{s_2^1(1 - s_2^1)(1 - 2 s_1^1) & \text{for } f^2_2.}
\end{cases} \quad (6)
\]

In the case \( P_2 \)'s interpretation function \( f_2 \) is fixed to \( f^1_2 \), there exist two asymptotically stable equilibrium points, \((s_1, s_2) = ([0, 1, 0]^T, [0, 1]^T) \) and \((0, 0, 1]^T, [1, 0]^T) \), and in the case that \( f_2 \) is fixed to \( f^2_2 \), there exists the unique asymptotically stable equilibrium point, \((s_1, s_2) = ([0, 1, 0]^T, [1, 0]^T) \).

In this paper, suppose that it is possible for \( P_2 \) to change the interpretation functions from \( f^1_2(s) \) (resp. \( f^2_2(s) \)) to \( f^1_2(s) \) (resp. \( f^2_2(s) \)). Hence, \( \Sigma = \{\sigma_1^{12}, \sigma_2^{21}\} \). From condition (3), we consider the following differences:

\[
\begin{align*}
R_2(f^2_2(s)) - R_2(f^1_2(s)) &= -s_1^2(3 - 2 s_2^2), \quad (7) \\
R_2(f^1_2(s)) - R_2(f^2_2(s)) &= s_1^2(3 - 2 s_2^1). \quad (8)
\end{align*}
\]

Figure 1 shows the contours of Eqs. (7) and (8). Since Eq. (7) is negative for all \( s_2^2 \) and \( s_1^1 \), the condition (3) can not be satisfied for all \( \lambda_{21} > 0 \). So the event \( \sigma_2^{12} \) can not occur, and \( P_2 \)'s interpretation function can not change from \( f^1_2 \) to \( f^2_2 \). All population states \( x \in S \) is feasible for \((f_1, f^1_2)\). However, population states \( x \in S \) satisfying Eq. (8) \( \geq \lambda_{21}^1 \) are infeasible for \((f_1, f^2_2)\). Figure 2 shows that an example of a transient behavior of Eqs. (4), (5), and (6). This example shows that \( P_2 \)'s initial interpretation \( f^1_2 \) changes to \( f^2_2 \) when Eq. (8) \( \geq \lambda_{21}^1 \) holds.

**Simulation II** Next, we change authorities’ payoffs of Soccer Hooliganism as shown in Table 2. In this case, authorities prefer the pair of “Acc.” and “Int” to that of “Unacc.” and “Int”.

Similar to Simulation I, replicator dynamics of \( P_1 \)'s strategies are formulated as Eqs. (4) and (5) in this situation. However, dynamics of \( P_2 \)'s strategies in this situation are different from that in Simulation I and given by

\[
\begin{align*}
\dot{s}_2^1 &= \begin{cases} 
2s_2^1 (1 - s_1^1)(1 - 2 s_1^1 - 2 s_2^1) & \text{for } f^1_2, \\
2s_2^1 (1 - s_2^1)(1 - 2 s_1^1) & \text{for } f^2_2.
\end{cases} \quad (9)
\end{align*}
\]

Multiplying Eq. (6) by two gives Eq. (9). Thus, Eqs. (6) and (9) have the same equilibrium points, and their stabilities are same.

Similar to Simulation I, suppose that \( \Sigma = \{\sigma_1^{12}, \sigma_2^{21}\} \).
Figure 3 shows the contours of Eqs. (10) and (11). In this situation, Eq. (10) is positive for $s_1^2 > 3/4$, and Eq. (11) is positive for $s_1^2 < 3/4$. Population states $s \in S$ satisfying Eq. (10) $< \lambda_2^1$ (resp. Eq. (11) $< \lambda_2^1$) are feasible for $(f_1, f_2^1)$ (resp. $(f_1, f_2^2)$). Figure 4 shows that an example of a transient behavior of Eqs. (4), (5), (9), and $P_2$’s interpretation.

From condition (3), we consider the following differences:

$$R_2 \left( f_2^1 (s) \right) - R \left( f_1^1 (s) \right) = -s_1^2 \left( 3 - 4s_2^1 \right). \quad \text{(10)}$$
$$R_2 \left( f_2^2 (s) \right) - R \left( f_2^2 (s) \right) = s_1^2 \left( 3 - 4s_2^1 \right). \quad \text{(11)}$$

Figure 3 shows the contours of Eqs. (10) and (11). In this situation, Eq. (10) is positive for $s_1^2 > 3/4$, and Eq. (11) is positive for $s_1^2 < 3/4$. Population states $s \in S$ satisfying Eq. (10) $< \lambda_2^1$ (resp. Eq. (11) $< \lambda_2^1$) are feasible for $(f_1, f_2^1)$ (resp. $(f_1, f_2^2)$). Figure 4 shows that an example of a transient behavior of Eqs. (4), (5), (9), and $P_2$’s interpretation.

5. Conclusions

In this paper, we have proposed replicator dynamics of hypergames with changes of players’ interpretations. We consider that each population has various interpretation functions and changes them depending on payoffs. Moreover, we have applied our proposed model to a well-known example of a hypergame “Soccer Hooliganism”, and we have shown that the interpretation changes depending on the evolutions of distributions of strategies. It is future work to discuss stabilities of interpretation functions and population states using stability theory of a hybrid system.

References


