A Realization of Determinant Criterion for STTC Design

Tatsuki FUKUDA, Shingo OTSU, Yuta TOKUNAGA and Hua-An ZHAO
Graduate School of Science and Technology, Kumamoto University
Kurokami 2-39-1, Kumamoto City, 860-8555, Japan
E-mail: tatsuki@st.cs.kumamoto-u.ac.jp

Abstract: In a MIMO (Multiple-Input Multiple-Output) system, space-time trellis coding (STTC) is one kind of space-time coding technique which can obtain both of high diversity and coding gain. In this paper, we propose a high-speed algorithm to solve the determinant calculations in determinant criterion of STTC. Some experimental results indicate that this algorithm is very available for developing an effective signaling scheme of space-time trellis codes.

1. Introduction

Multiple-input multiple-output (MIMO) systems have better performance under fading channels and can provide high data rates and high diversity for satisfying the modern wireless communications demands. A MIMO system where multiple transmit antennas $N_T$ and receive antennas $N_R$ are used, has been adopted as next generation (4G) technology in wireless communications. Because a MIMO system emerges a lot of new technologies such as space-time coding and decoding so that it makes wireless communications an exciting and challenging field. The main features of MIMO systems are broadband and diversity techniques, the former provides high data transmission rate and the latter realizes the better signal qualities in fading channel environments. Also, the low power consumptions are strictly required.

The space-time (ST) coding can make good use of the transmit diversity to alleviate the effect of fading and achieve high date transmission rate without adding extra bandwidth and power consumption. Hence, to develop good ST codes is very important. The first ST code with a normalized rate of 1 symbol per channel use (spcu) was proposed by Alamouti over two transmit antennas and two time periods in [1]. In [2], Tarokh gave the design criteria, which is a tradeoff between constellation size, data rate, complexity, and diversity advantage for ST code, and he presented STTC. By expanding the Alamouti code, a lot of researches for the new type ST codes called ST block code (STBC) were presented. STBC can achieve the maximum possible diversity advantage with a simple decoding algorithm. However, no coding gain can be provided by STBC. However, STTC is able to combat the effect of fading, which can simultaneously offer a substantial coding gain, spectral efficiency, and diversity improvement on flat fading channels.

In order to develop good STTC, the rank criterion and the determinant criterion are proposed [2], [3]. The rank criterion guarantees to achieve maximum possible diversity $N_T N_R$ if matrix $A(c, e)$ (defined by Eq.(5)) has full rank for all the codewords, and the determinant criterion can achieve maximum coding gain if the minimum determinant of $A(c, e)$ should be maximized over all codewords.

However, how to calculate $\det(A(c, e))$ has not been satisfied in the determinant criterion. Generally, the calculation of $\det(A(c, e))$ is realized by the combinations of all $c$ and $e$. Thus, to find an effective method to calculate $\det(A(c, e))$ is very important. In this paper, we present an high-speed algorithm to calculate $\det(A(c, e))$ so that the determinant criterion can be realized easily.

2. MIMO Systems and STTC Structures

We consider a MIMO system with $N_T$ transmit antennas and $N_R$ receive antennas shown as Fig.1, where the STC is a space-time coder and the STD is a space-time decoder. In the STC, the serial binary inputs are changed into the parallel symbol vector $X_t = (x_1^t, x_2^t, \ldots, x_{N_T}^t)^T$ and the received signal matrix $Y_t$ can then be written as

$$Y_t = HX_t + N_t,$$

where $H$ is a $N_R \times N_T$ channel matrix whose entries are independent, identically distributed (i.i.d.) complex Gaussian random variables with variance 0.5 in each dimension, $i.e., h_{j,t} \sim \mathcal{CN}(0,1)$; $N_t = (n_1^t, n_2^t, \ldots, n_{N_T}^t)^T$ is a noise matrix with i.i.d. complex Gaussian entries. In this paper, we assume the channels to be quasi-static and flat Rayleigh fading.

The STTC is one of signal processing techniques and can improve the transmit performance of a MIMO system. Let $(a_i, b_i)$ be binary inputs at time $t$ and make $u_t = (a_1, b_1, a_2, b_2, \ldots, a_{\text{MPSK}}, b_{\text{MPSK}})$ where $s$ denotes the number of memory elements in the encoder, and $G$ be a generation matrix with $N_T$ columns and $M+s$ rows where $m$ is the number of binary inputs at time $t$. The STTC with $M$-PSK modulation is given by

$$c = (c_1, c_2)^T = u_tG \pmod{M}. \quad (2)$$

The transmitted signals can be obtained by $X_t = (x_1^t, x_2^t)^T = (\exp(j\pi c_1^t/2), \exp(j\pi c_2^t/2))^T$.

For example, the generation matrix $G$ introduced by Tarokh in [2] is

![Figure 1: A MIMO system]

The STTC with $M$-PSK modulation is given by

$$c = (c_1, c_2)^T = u_tG \pmod{M}. \quad (2)$$

The transmitted signals can be obtained by $X_t = (x_1^t, x_2^t)^T = (\exp(j\pi c_1^t/2), \exp(j\pi c_2^t/2))^T$.

For example, the generation matrix $G$ introduced by Tarokh in [2] is

$$G = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 0 & \ldots & 1
\end{bmatrix}.$$
and the STTC structure can be represented by a trellis diagram shown as in Fig.2, which indicates the principle of the encoder of STTC. Fig.2 shows the four-state STTC structure where the nodes of q_{0,0}, q_{0,1}, q_{1,0}, and q_{1,1} represent the states, the edges represent state transitions, and the sequences of the number represent the codeword transmitted from antennas. When the binary input 10 are input at state of q_{0,1}, the transition edge is 10, the output is (1, 2), and the next state will be q_{1,0}. It is not difficult to make sure the output c of STTC are (0, 0), (0, 1), (1, 1), (2, 2), (2, 0) with the initial state q_{0,0} when binary input sequence is 00→11→01→10→00 in Fig.2. It should be noted when we create the transmitted symbol, both of the first state and the last state should be q_{0,0}. We can use the trellis diagram to create and detect the codewords.

![Figure 2: Trellis diagram for a generation matrix by Tarokh](image)

### 3. The Proposed Algorithm

Traditionally, the design of STTC is treated as a problem of minimizing the probability of error for a fixed rate and SNR.

#### 3.1 Pair-wise Error Probability

In the quasi-static fading model, the Chernoff bound of the pair-wise error probability [3] of deciding in favor of an erroneous symbol vector e when transmitting c is as follows

\[ P_{c \rightarrow e} = \text{det}(I_{N_t} + \rho A(c, e))^{-N_t} \]  

(4)

where \( I_{N_t} \) is an \( N_t \times N_t \) identity matrix; \( \rho \) denotes the average SNR at each receive antenna; \( A(c, e) \) is the distance matrix given by

\[ A(c, e) = B(c, e) B^*(c, e) \]  

(5)

where

\[ B(c, e) =  
\begin{bmatrix}
    c_1 & e_1 & e_2 & \cdots & e_L & e_L^* \\
    c_1^2 & e_2 & e_2^2 & \cdots & e_L^2 & e_L^2^* \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    c_1^{N_c} & e_2 & e_2^{N_c} & \cdots & e_L^{N_c} & e_L^{N_c}^* \\
\end{bmatrix} \]  

(6)

and \( c_i \) and \( e_i \) denote the transmitted symbol and the erroneous detected symbol from antenna \( i \) at time \( t \), the \( L \) and \( N_t \) denote the length of frame and the number of transmit antennas. At high SNR, Eq.(4) leads to the rank criterion and the determinant criterion for STTC designs where the rank and the determinant of \( A(c, e) \) must be calculated.

Here, we consider how to calculate \( \text{det}(A(c, e)) \) by an effective method. By Eq.(5), we know that the \( A(c, e) \) can be represented by

\[ A(c, e) = \sum_{i=1}^{L} b_i b_i^* \]  

(7)

\[ b_i = [c_1^i - e_1^i, c_1^2 - e_1^2, \cdots, c_1^{N_c} - e_1^{N_c}]^T \]  

(8)

The question is how to obtain the minimum \( \text{det}(A(c, e)) \) with all combination of the codewords \( c \) and \( e \). From Eq.(7), we know that all of \( \text{det}(A(c, e)) \) can be obtained by \( \text{det}(\sum_{i=1}^{L} b_i b_i^*) \). It is evident that the calculation of \( \text{det}(A(c, e)) \) is based on the combinations of all \( c \) and \( e \).

#### 3.2 State Diagram Representation

By Eq.(7), we know that the determinant can be calculated by means of adding \( b_i b_i^* \) which consist of \( c \) and \( e \) generated by each state transition of the trellis diagram.

For example, we consider the calculation of the minimum determinant of \( G \) represented by Eq.(3). Without loss of generality, we assume that \( c \) is the all zero codeword and we can get the state diagram represented by Fig.3 [2].

![Figure 3: State diagram for a generation matrix by Tarokh](image)
The nodes represent the states, and the nodes at the depth \( t \) represent the state at time \( t \). All of the nodes are labeled by number like \( \text{"i"}. \) The node labeled \( \text{"i"} \) means that the \( j \)-th node at the depth \( i \), and we indicate the node as \( N_{i,j} \) from here. The root node represents the initial state. Each node has four child nodes because there should be four cases of detecting. Each node has the matrix \( D \) which can be obtained as follows. For a node \( N_{i,j} \) whose parents have the matrix \( D_{\text{parent}} \), the matrix \( D_{\text{child}} \) of \( N_{i,j} \) is calculated by \( D_{\text{parent}} + b_i^* b_j^* \), where the \( b_i^* \) is the matrix represented on the edge which connects the parent node of \( N_{i,j} \) to \( N_{i,j} \) in the state diagram. Note that we assume the root node matrix \( D_{\text{root}} \) is a zero matrix. The tree structure has the following property.

**Property 1:** The distance matrices as Eq. (7) for all \( e \) and \( c \) are identical with the matrices of the nodes at the depth \( L \).

From **Property 1**, the matrix having the minimum determinant exists in all matrices \( D \) of the nodes at the depth \( L \). On the other hand, if a node \( N_{i,j} \) whose state is \( q_{0,0} \) has \( D_{\text{parent}} \), there is a matrix \( D_{\text{child}} \) which is identical to \( D_{\text{parent}} \), because there is a path between \( N_{i,j} \) and \( N_{\text{root}} \) surely whose node states are all of \( q_{0,0} \) and all edges have zero matrices. It means that we can obtain the minimum determinant of the distance matrix by (1) searching nodes whose states are \( q_{0,0} \), (2) finding a minimum determinant in all determinants of such nodes obtained by (1).

**Property 2:** Blum proved the following Eq. (9) [8].

\[
\det(\sum_{r=1}^{k} b_r b_r^*) \leq \det(\sum_{r=1}^{k-1} b_r b_r^*)
\]

(9)

Eq. (9) means that the determinant value doesn’t get smaller when adding the matrix \( b_i^* b_i^* \), so we have the following Reducing Rules. By these rules, a lot of the redundancy nodes can be eliminated so that we can calculate the minimum determinant quickly. In fact, these rules enable the calculation to be finished at most at the depth as the trellis states number in the tree structure.

**Reducing Rule 1:** When we obtained a minimum determinant as a candidate, we can eliminate all nodes whose determinants are not less than the candidate.

**Reducing Rule 2:** Let a node \( N_2 \) be the descendant node of node \( N_1 \). If both \( N_1 \) and \( N_2 \) have the same states, we can eliminate all descendant nodes of \( N_2 \).

### 3.4 High-Speed Algorithm

In section 3.3, we know the calculation of the minimum determinant can be realized by making tree structure. Here, we propose a high-speed algorithm for calculating the minimum determinant of the distance matrix based on the tree structure. The outline of the algorithm is as follows.

First, we assume an initial value of the minimum determinant “\( c_1 \)” to be infinity and the parent node set \( N_p \) to contain the root node. Next, we make the child nodes of the node(s) in \( N_p \) and make a child node set \( N_c \) contain these child nodes. Then, we calculate the determinants of the nodes in the \( N_c \). Second, we choose the one node denoted by \( N_{\text{act}} \) by ascending order of the determinant value from \( N_c \). We deal with \( N_{\text{act}} \) by one of the following processing (1) – (4) in numerical order.

1. By **Reducing Rule 2**, we eliminate the \( N_{\text{act}} \) if the state of \( N_{\text{act}} \) has already existed in the path from the root node to \( N_{\text{act}} \).
2. By **Reducing Rule 1**, we eliminate the \( N_{\text{act}} \) and the nodes in \( N_c \) which have not chosen yet if the determinant of \( N_{\text{act}} \) is not less than \( c_1 \).
3. By **Property 1**, we renew candidate with the determinant value of \( N_{\text{act}} \) if the state of \( N_{\text{act}} \) is the same as \( q_{0,0} \), and we eliminate the \( N_{\text{act}} \) and the nodes in \( N_c \) which have not chosen yet because of **Reducing Rule 1**.
4. Otherwise, we do nothing.

After finishing the processing (1)–(4), we chose the next node as \( N_{\text{act}} \) in \( N_c \) and repeat (1)-(4) again until all of the nodes in \( N_c \) have been chosen. If there are some nodes in \( N_c \), we change \( N_c \) into \( N_p \) and we do the same processing.

By above processing, we can make the tree structure. It can be seen that the depth of tree does not exceed the number of its trellis states.

According to the above explanations, we propose a high-speed algorithm for calculating the minimum determinant.

---

**Algorithm for calculating \( \det(\mathbf{A}(e, e)) \):**

**Step 1:** Let \( \text{candidate} \) be infinity, and put the root node in the set \( N_p \).

**Step 2:** Make a child node set \( N_c \) of the nodes in \( N_p \).

**Step 3:** Choose one node \( N_{\text{act}} \) in ascending order from \( N_c \).

**Step 4:** If the state of \( N_{\text{act}} \) has already existed in the path from the root node to \( N_{\text{act}} \), eliminate the \( N_{\text{act}} \) and go to Step 7.

**Step 5:** If the determinant of \( N_{\text{act}} \) is not less than the minimum candidate, eliminate the \( N_{\text{act}} \) and the nodes in \( N_c \) which have not chosen yet, and go to Step 7.

**Step 6:** If the state of \( N_{\text{act}} \) is \( q_{0,0} \), renew the minimum candidate with the determinant of \( N_{\text{act}} \), and eliminate the \( N_{\text{act}} \) and the nodes in \( N_c \) which have not been chosen yet.

**Step 7:** Choose the next node in \( N_c \) and go to Step 4.
Step 8: If there’s no node in $N_c$, output the minimum candidate and finish the calculation.

Step 9: Change the $N_c$ into $N_p$ and go to Step 2.

The proposed algorithm is in the assumption for fixing $c$ as zero codeword. When $c$ is not fixed, this algorithm also can be used by the following extensions. (1) Each node in the tree structure represents a pair of both states of detecting and coding. (2) In Step 6, check if the states of a pair indicated by $N_{act}$ are equal to each other.

4. Performance of the Algorithm

To examine our algorithm’s performance, we made experiments with the 20000 generation matrices with 4 rows and 2 columns, and 20000 ones with 5 rows and 2 columns. We examined how deeply the tree is when the calculation finished. The depth of finishing calculation is three if all of the nodes in $N_c$ can eliminate at the depth three. The results which are the ratio of the number of the generation matrices finishing calculation at each depth are shown in Table 1 where you can find that a lot of the calculation can finish in the depth of three or four and the maximum depth is the same number as the trellis states. It follows that this algorithm can calculate the minimum determinant for almost all of the generation matrices.

Table 1: The depth finishing the calculation

<table>
<thead>
<tr>
<th>The depth finishing calculation</th>
<th>The size of generation matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4 x 2</td>
</tr>
<tr>
<td>2</td>
<td>60.295 %</td>
</tr>
<tr>
<td>3</td>
<td>32.96 %</td>
</tr>
<tr>
<td>4</td>
<td>6.745 %</td>
</tr>
<tr>
<td>5</td>
<td>2.435 %</td>
</tr>
<tr>
<td>6</td>
<td>1.715 %</td>
</tr>
<tr>
<td>7</td>
<td>0.05 %</td>
</tr>
<tr>
<td>8</td>
<td>0.05 %</td>
</tr>
<tr>
<td></td>
<td>5 x 2</td>
</tr>
<tr>
<td>2</td>
<td>17.35 %</td>
</tr>
<tr>
<td>3</td>
<td>42.255 %</td>
</tr>
<tr>
<td>4</td>
<td>26.72 %</td>
</tr>
<tr>
<td>5</td>
<td>9.52 %</td>
</tr>
<tr>
<td>6</td>
<td>2.435 %</td>
</tr>
<tr>
<td>7</td>
<td>1.715 %</td>
</tr>
<tr>
<td>8</td>
<td>0.05 %</td>
</tr>
</tbody>
</table>

For verifying this algorithm, we calculated the minimum determinants of the distance matrices for the generation matrices proposed up to now. The calculating time using MATLAB are shown in Table 2 where the generation matrix proposed by Meixia [7] is calculated by the extended algorithm which $c$ is not fixed. It can be seen that all of the calculations finished in less than 1.5 ms.

Table 2: The time to calculate the minimum determinants

<table>
<thead>
<tr>
<th>Generation matrix</th>
<th>Proposer</th>
<th>Time[ms]</th>
</tr>
</thead>
</table>
| 0 0 2 1
| 2 1 0 0          | Tarokh, et al.[2]    | 1.3281   |
| 0 2 1 2
| 2 1 2 0          | Meixia, et al.[7]    | 1.4531   |
| 0 2 1 2
| 2 3 2 0          | Bahador, et al.[9]   | 1.3750   |
| 2 0 1 3
| 2 2 0 1          | S. Baro, et al.[6]   | 1.3438   |
| 3 2 2 1
| 3 2 1 1          | T. INOUE, et al.[10] | 1.3281   |

Generally, when we develop a new STTC based on the determinant criterion, the minimum determinants of all generation matrices must be calculated. This high-speed algorithm is very useful for STTC design.

Acknowledgment

The author would like to thank all of Liu research group of Nanjing University of Posts and Telecommunications, China, for their simulation supports.

References