A Mathematical Framework for Propagation in an Open Cavity

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Abstract—The purpose of this paper is to establish a formal mathematical framework for the electromagnetic wave propagation in an arbitrary cavity. The walls of the cavity are not assumed perfectly conducting and we use the transmission boundary conditions for the tangential fields. Our main tool is the Laplace transform. The focus here is on the modeling and detailed proofs or calculations are not provided.

I. INTRODUCTION

Our objective in this short note is to provide a formal and working mathematical reasoning for the so called transmission problem in time domain electromagnetics (E/M). We treat the case of an open cavity i.e., a bounded domain in \( \mathbb{R}^3 \) which is penetrable by the E/M field. As boundary conditions we assume the tangential continuity of both electric and magnetic field. This is a realistic assumption for many problems in the “real world”. The case of propagation inside a perfect conducting cavity is treated in detail in [1]; see also [2] for the corresponding spectral problem in the frequency domain. The interior E/M transmission problem has been studied, among others, in [3], [4]. For a rather complete contemporary account of mathematical methods in E/M problems we refer to [5].

From our point of view, we are especially interested in E/M propagation and scattering problems in and from certain open structures like HDVC power cables, e.g., see [6], [7]. In these references, the power cable is modeled as a multilayered circular waveguide and the problem is treated by using cylindrical vector waves and complex analysis techniques.

The investigation of the relevant spectral properties possess a central role in our research. In the frequency domain, an eigenvalue (with a corresponding eigenvector) represents an eigenfrequency of the cavity (with a corresponding mode). But, in contrast with the perfect conducting walls, there exist frequencies that are not eigenfrequencies. In the aforementioned references, the propagation constants (counterparts of eigenfrequencies in waveguides) are characterized as poles of an analytic function in the complex plane, known as the dispersion relation. The values that are not propagation constants appear as branch-cuts of the same function. An account for the mathematical formulation of the eigenvalue problem and the characterization of the modes in the case of an optical fiber is given in [8]. For computational aspects of the relevant problem, one may refer to [9].

II. STATEMENT OF THE PROBLEM

Let \( \Omega^- \subset \mathbb{R}^3 \) be an open bounded set. We denote by \( \Omega^+ := \mathbb{R}^3 \setminus \Omega^- \) the exterior of \( \Omega^- \). We assume that \( \Omega^- \) is occupied by vacuum. A typical point in \( \mathbb{R}^3 \) will be denoted by \( \mathbf{r} \) and its modulus by \( r \). The boundary \( \Gamma := \partial \Omega^- = \partial \Omega^+ \) is assumed to be smooth enough and the outward normal \( \mathbf{n} \) is defined on \( \Gamma \) and points inside \( \Omega^+ \).

The electromagnetic wave propagation in \( \Omega^- \), resp. in \( \Omega^+ \), is described by four time-dependent vector fields \( \mathbf{D}^-, \mathbf{B}^-, \mathbf{E}^-, \mathbf{H}^- \), resp. \( \mathbf{D}^+, \mathbf{B}^+, \mathbf{E}^+, \mathbf{H}^+ \) (functions of \( t \geq 0 \) and \( \mathbf{r} \in \Omega^\pm \)), representing the electric flux, the magnetic induction, the electric field and the magnetic field in both domains. The fields satisfy the source free Maxwell equations

\[
\begin{align*}
\frac{\partial \mathbf{D}^\pm}{\partial t} &= \nabla \times \mathbf{H}^\pm, \quad \text{in } \Omega^\pm, \quad (1) \\
\frac{\partial \mathbf{B}^\pm}{\partial t} &= - \nabla \times \mathbf{E}^\pm, \quad \text{in } \Omega^\pm.
\end{align*}
\]

The Gauss equations state that the fields \( \mathbf{D} \) and \( \mathbf{B} \) are divergence free

\[
\text{div } \mathbf{D}^\pm = \text{div } \mathbf{B}^\pm = 0, \quad \text{in } \Omega^\pm. \quad (2)
\]

They are valid for every time instant if this happens for one specific time instant.

On the boundary we have the tangential continuity of the fields [10, pp. 7–8]

\[
\begin{align*}
\mathbf{n} \times \mathbf{E}^- &= \mathbf{n} \times \mathbf{E}^+, \quad \text{on } \Gamma. \\
\mathbf{n} \times \mathbf{H}^- &= \mathbf{n} \times \mathbf{H}^+.
\end{align*}
\]

(3)

The exterior fields satisfy the Silver-Müller radiation condition [11, p. 113] (boundary condition at infinity)

\[
\lim_{r \to \infty} (\mathbf{E}^+ \times \mathbf{r} + r \mathbf{H}^+) = 0. \quad (4)
\]

Initial conditions are also provided

\[
\mathbf{D}^\pm(0, \mathbf{r}) = \mathbf{D}_0^\pm(\mathbf{r}), \quad \mathbf{B}^\pm(0, \mathbf{r}) = \mathbf{B}_0^\pm(\mathbf{r}). \quad (5)
\]

Observe that, by hypothesis, \( \mathbf{D}_0^\pm, \mathbf{B}_0^\pm \) are divergence-free.

The constitutive relations for the interior fields obey a causal, time invariant model [12, Ch. 2]

\[
\begin{align*}
\mathbf{D}^- &= \varepsilon_0 \varepsilon_r \mathbf{E}^- + \varepsilon_0 \int_0^t \chi(t-s) \mathbf{E}^-(s) \, ds, \quad \mathbf{B}^- = \mu_0 \mathbf{H}^- .
\end{align*}
\]

(6)
For the exterior fields the constitutive relations are
\[ D^+ = e_0 E^+ , \quad B^+ = \mu_0 H^+ . \] (7)
\( e_0, \mu_0 \) are positive or constants, \( e_\pm \) is a real (resp. complex) and \( \chi \) is a well behaved real (resp. complex) function in the case of a lossless (resp. lossy) medium. It is worth noting that initial conditions could be given for fields \( E \) and \( H \).

### III. LAPLACE TRANSFORM
We apply formally the Laplace transform [13] in (1) and, with the aim of (5), (6), (7),
\[ \lambda (\varepsilon_r + \tilde{\chi}(\lambda)) e_0 \hat{E}^-(\lambda) - \text{curl} \hat{H}^-(\lambda) = D_0^-, \]
\[ \lambda \mu_0 \hat{H}^-(\lambda) + \text{curl} \hat{E}^-(\lambda) = B_0^- , \]
\[ \lambda \varepsilon_0 \hat{E}^+(\lambda) - \text{curl} \hat{H}^+(\lambda) = D_0^+ , \]
\[ \lambda \mu_0 \hat{H}^+(\lambda) + \text{curl} \hat{E}^+(\lambda) = B_0^+ . \] (8)
Another way to obtain (8) is to apply a standard separation-of-variables argument and search for solutions of (1) of the form
\[ E^\pm (r,t) = \text{Re} [\hat{E}^\pm (\lambda) e^{-\lambda t}] H(t), \]
\[ H^\pm (r,t) = \text{Re} [\hat{H}^\pm (\lambda) e^{-\lambda t}] H(t), \]
where
\[ H(t) := \begin{cases} 1 , & t > 0, \\ 0 , & t < 0 \end{cases} \]
is the Heaviside function. By (8b), (8d) one calculates the \( H \)-field by the \( E \)-field
\[ \hat{H}^\pm (\lambda) = \frac{1}{\lambda \mu_0} \left( \hat{B}_0^\pm \text{curl} \hat{E}^\pm (\lambda) \right) . \] (9)

We will usually suppress \( \lambda \) for notational convenience. By substituting (9) to (8a), (8c), we obtain
\[ \text{curl curl} \hat{E}^- + \kappa^2 \eta \hat{E}^- = \lambda \mu_0 D_0^- + \text{curl} B_0^- , \]
\[ \text{curl curl} \hat{E}^+ + \kappa^2 \hat{E}^+ = \lambda \mu_0 D_0^+ + \text{curl} B_0^+ , \] (10)
where \( \kappa^2 := \lambda^2 e_0 \mu_0, \eta := \varepsilon_r + \tilde{\chi}(\lambda) \). The vector identity
\[ \text{curl curl} \mathbf{A} = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) \]
holds for any vector field \( \mathbf{A} \) in \( \mathbb{R}^3 \). Thanks to (2), (6), (7), we imply that both \( \hat{E}^\pm \) are divergence free and consequently equations (10) read
\[ -\nabla^2 \hat{E}^- + \kappa^2 \eta \hat{E}^- = \lambda \mu_0 D_0^- + \text{curl} B_0^- , \]
\[ -\nabla^2 \hat{E}^+ + \kappa^2 \hat{E}^+ = \lambda \mu_0 D_0^+ + \text{curl} B_0^+ . \] (11)
Under certain conditions, as it is stated in [11, Th. 4.3], equations (9), (11) are equivalent to (8); more precisely, if \( \hat{E}^\pm \) are divergence-free solutions to (11) then, along with (9), they provide solution to (8). We therefore stick to (11), which form a system of two vector Helmholtz equations, coupling at boundary conditions on \( \Gamma \), which now read
\[ n \times \hat{E}^- = n \times \hat{E}^+ , \]
\[ n \times \left( B_0^- - \text{curl} \hat{E}^- \right) = n \times \left( B_0^+ - \text{curl} \hat{E}^+ \right) . \] (12)
The radiation condition is written
\[ \lim_{r \to \infty} \left[ \hat{E}^+ \times r + r \left( B_0^+ - \text{curl} \hat{E}^+ \right) \right] = 0. \] (13)
The problem (11), (12), (13) is a version of the dielectric transmission problem for Maxwell system and the corresponding vector Helmholtz equation; this problem has been studied in various forms, see [14] and the references therein. The difficulty here is the presence of sources in the right hand sides, as they are provided by the initial conditions. This means that any method which involves integral representations has to introduce not only surface but volume integral operators as well.

### IV. ON THE METHOD OF SOLUTION
Our model problem can be formulated as follows: suppose we are given a function \( f \) defined on \( \mathbb{R}^3 \setminus \Gamma \) and functions \( b^\pm \) defined on \( \Gamma \). We write
\[ f(r) = \begin{cases} f^-(r) , & r \in \Omega^-, \\ f^+(r) , & r \in \Omega^+. \end{cases} \]
We also define
\[ k = k(r) := \begin{cases} \kappa r - \eta , & r \in \Omega^-, \\ \kappa r , & r \in \Omega^+. \end{cases} \]
where \( \kappa, \eta \) are given complex numbers. Now, find a divergence-free function \( u \) defined on \( \mathbb{R}^3 \) satisfying the Helmholtz equation in \( \mathbb{R}^3 \setminus \Gamma \),
\[ \nabla^2 u - k^2 u = f, \] (14)

and the boundary conditions on \( \Gamma \),
\[ n \times u^- = n \times u^+, \]
\[ n \times (b^- - \text{curl} u^-) = n \times (b^+ - \text{curl} u^+) . \] (15)
As usual, \( u^\pm \) denotes the restriction of \( u \) in \( \Omega^\pm \). \( u^+ \) satisfies the radiation condition
\[ \lim_{r \to \infty} \left[ u^+ \times r + r \left( b^+ - \text{curl} u^+ \right) \right] = 0. \] (16)
We now use a trick, following Remark 4 in [15], to rewrite the problem in question as an integral equation. Indeed, (14) is read
\[ \nabla^2 u - k^2 u = -k^2 \eta u + f , \text{ in } \mathbb{R}^3 \setminus \Gamma , \]
with the agreement that \( \eta \) vanishes in \( \Omega^+ \). Let \( G_\kappa(r) \) be the divergence-free fundamental solution of the vectorial Helmholtz operator \( \nabla^2 - k^2 \). That is to say, \( G_\kappa \) is a \( 3 \times 3 \) function matrix such that
\[ \nabla^2 G_\kappa - k^2 G_\kappa = \delta I \] (18)
and
\[ \text{div} G_\kappa = 0. \] (19)
in the sense of distributions.
\( G_\kappa \) can be represented as follows,
\[ G_\kappa(r) := g_\kappa(r) I + \frac{1}{k^2} H g_\kappa(r) , \] (20)
where
\[ g_\kappa(r) := \frac{e^{-\kappa r}}{4\pi r^2} \]  
(21)
is the fundamental solution of the corresponding scalar Helmholtz operator \( \nabla^2 - \kappa^2 \) and \( H := \nabla^2 \nabla \) is the Hessian matrix. Note that, for \( \Re \kappa > 0 \), (21) defines an outgoing solution \( i.e., \) it satisfies the Sommerfeld radiation condition [11, p. 69].

Then, by (17), we obtain formally the integral equation
\[
\begin{align*}
\mathbf{u}(r) &= -\kappa^2 \int_{\mathbb{R}^3} \frac{\eta(r')}{r} \mathbf{G}_\kappa(r-r')\mathbf{u}(r') \, dr' + \\
&\quad + \int_{\mathbb{R}^3} \mathbf{G}_\kappa(r-r')\mathbf{f}(r') \, dr', \quad r \in \mathbb{R}^3.
\end{align*}
\]
(22)
But, as already stated, \( \eta \) vanishes outside \( \Omega^- \) and the first integral in the right hand side of (22) is taken merely on \( \Omega^- \). This fact enables us to define the volume integral operator
\[
\mathbf{(}\mathbf{A}_\kappa\mathbf{v})(r) := \int_{\Omega^-} \mathbf{G}_\kappa(r-r')\mathbf{v}(r') \, dr', \quad r \in \mathbb{R}^3,
\]
and the known function
\[
\mathbf{d}(r) := \int_{\mathbb{R}^3} \mathbf{G}_\kappa(r-r')\mathbf{f}(r') \, dr', \quad r \in \mathbb{R}^3.
\]
and write (22) as follows
\[
\mathbf{u} = -\kappa^2 \eta A^* \mathbf{u} - \mathbf{d}.
\]
(23)
By restriction in \( \Omega^\pm \), (23) gives the pair of equations
\[
\begin{align*}
\mathbf{u}^- &= -\kappa^2 \eta A^* \mathbf{u}^- + \mathbf{d}^-, \quad (24a) \\
\mathbf{u}^+ &= -\kappa^2 \eta A^* \mathbf{u}^+ + \mathbf{d}^+.
\end{align*}
\]
(24b)
Then (24a) is a volume integral equation of second kind, from which we can calculate \( \mathbf{u}^- \) as
\[
\mathbf{u}^- = (I + \kappa^2 \eta A^*)^{-1} \mathbf{d}^-.
\]
(25)
In the sequel, we can substitute to (24b) and calculate \( \mathbf{u}^+ \).

Let us now check the boundary conditions (15). It is evident that the conditions are just transferred to function \( \mathbf{d} \) \( i.e., \) (15) are valid if and only if
\[
\begin{align*}
\mathbf{n} \times \mathbf{d}^- &= \mathbf{n} \times \mathbf{d}^+, \quad (26a) \\
\mathbf{n} \times (\mathbf{b}^+ - \operatorname{curl} \mathbf{d}^-) &= \mathbf{n} \times (\mathbf{b}^+ - \operatorname{curl} \mathbf{d}^+). \quad (26b)
\end{align*}
\]
Moreover, (26a) apply restrictions on the traces of \( \mathbf{f}^\pm \) or, equivalently, restrictions on the traces of \( \mathbf{f} \) form outside and inside the cavity, respectively.

The radiation condition (16) is satisfied provided that
\[
\lim_{r \to \infty} r \mathbf{b}^+ = \mathbf{0} \quad \text{e.g., if } \mathbf{b}^+ \text{ has compact support. This fact, together with (26b) apply restrictions on the boundary data } \mathbf{b}^\pm.
\]
\(\sigma_\tau \neq \emptyset\) (e.g., \(X\) is a Hilbert space and \(A\) is self-adjoint), then for \(\mu \in \sigma(A)\) just one of the following is possible:

- \(\mu\) is an isolated eigenvalue with a finite dimensional eigenspace, that is \(\mu \in \sigma_d(A)\).
- \(\mu\) is an eigenvalue wich is not isolated or with an infinite dimensional eigenspace, that is \(\mu \in \sigma_{ess}(A) \cap \sigma_p(A)\).
- \(\mu\) lies on the continuous spectrum, that is \(\mu \in \sigma_c(A)\).

It should be stated here that there are various definitions in the literature for the concept of essential spectrum, which may or may not coincide. For a relevant discussion for Hilbert space case, we refer to [19].

**Definition 4.** A complex number \(\kappa\) is called a regular value of \(P(\cdot)\) if \(P(\kappa)^{-1} \in B(X)\). The set of regular values of \(P(\cdot)\) is called the resolvent set and is denoted by \(\rho(P(\cdot))\). The complement \(\sigma(P(\cdot)) := \mathbb{C}/\rho(P(\cdot))\) is called the spectrum of \(P(\cdot)\).

**Corollary 1.** The following are equivalent:

- \((a)\) \(\kappa \in \sigma(P(\cdot))\),
- \((b)\) \(\kappa \in \sigma(\Lambda_\kappa)\).

The above corollary provide us a method to calculate the spectrum of \(P(\cdot)\) by using the knowledge of the spectrum of \(\Lambda_\kappa\), which depends on \(\kappa\).

**Proposition 1.** Let \(\kappa \neq 0\), \(\kappa \in \sigma(P(\cdot))\) if and only if there exists \(\mu(\kappa) \in \sigma(\Lambda_\kappa)\) such that \(\kappa\) is a solution of the equation

\[
\mu(\kappa)\eta \kappa^2 = 0. \tag{28}
\]

Equation (28) deserves a special name and we refer to it as the dispersion relation. Definition 4, Corollary 1 and Proposition 1 are formulated in an exactly similar way for the point, continuous, residual, essential and discrete spectra.

**VI. THE E/M TRANSMISSION PROBLEM**

The abstract model described in sections IV, V can be used to solve the problem (11), (12), (13). More precisely, we set

- The unknown \(u\) to be the electric field \(\mathbf{E}\).
- The current \(f\) to be \(\lambda \mathbf{B}_0 + \text{curl} \mathbf{B}_1\).
- The “boundary data” \(b^\pm\) to be the traces \(\mathbf{B}_0^\pm\) of \(\mathbf{B}_0\) from outside and inside, respectively.
- \(\kappa := \lambda \sqrt{\varepsilon_0 \mu_0}, \eta := \varepsilon_\tau + \chi(\lambda)\).

The spectral theoretic properties are transferred from the parameter \(\kappa\) to \(\lambda\). Namely, the problem is well-posed for \(\lambda \sqrt{\varepsilon_0 \mu_0} \in \rho(P(\cdot))\) and the transformed electric field \(\mathbf{E}(\lambda)\) is then calculated by (25). The corresponding magnetic field \(\mathbf{H}(\lambda)\) is then calculated by (9).

Actually, one needs to calculate the transformed E/M field for a vertical line in the complex plane of the form \(a + i\mathbb{R}\) and then invert the Laplace transform to get back in the time domain. In this way we obtain the original E/M field \(\mathbf{E}(t), \mathbf{H}(t)\) or, in other words, we construct the semigroup of the evolution problem.

**VII. CONCLUSION**

In this paper we consider the electromagnetic transmission problem in the time domain for a penetrable cavity and with discontinuous material parameters. By using the Laplace transform, we reformulate the problem as a volume integral equation and we discuss its solvability via spectral theoretic arguments. Detailed proofs or calculations are not provided, since we focus mainly on modeling and the development of general machinery.

**REFERENCES**


