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Information Processing in Fractally Coupled Networks

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Abstract—Currently, it is believed that synchronization plays an important role in sensory information processing in the primary visual cortex. It also has been proposed that cortical neural networks develop under the constraint of minimization of the total length of connections. We study synchronization behavior, and information transfer, in models of neural networks with variable architectures. The investigation reveals that fractally coupled networks with a bi-power law distribution of connections perform best under the above constraints.

1. Introduction

The primary visual pathway relays information from the retina across the lateral geniculate nucleus to the striate cortex. The morphology of this cortex exhibit a vertical and a horizontal structure. The former divides the cortex into layers, that process anatomically distinct, parallel inputs from the retina [2]. In the cat, layer IVC α cells are believed to possess orientation selectivity. These cells respond best to a thin line that is aligned with the main axis of the receptive field. A poor response is observed when the stimulus is perpendicular to this axis. In the horizontal plane, different columns can be distinguished. Although the inputs from the retina arrive by independent pathways, the preferred orientation of neurons in different layers of a column is the same. Therefore, these columns are called orientation columns; together, they build up a two dimensional orientation map. Recurrent connections to areas of higher visual processing embed the striate cortex within whole cortex.

An extended object activates a substantial portion of the visual field, thus neural activity is distributed across many different cortical columns, where the edges of the object are encoded at different locations of the orientation map. The binding hypothesis states that neurons that respond to features of the same object, increase synchrony [12]. Whereas some studies appear to support this binding-by-synchronization hypothesis [13], its validity, and in particular the reproductibility of the relevant key experiments remain heavily disputed [11, 14, 9].

Numerous constraints impose themselves on the organization of cortical networks. The global connectivity needs to be very sparse, in order to reduce the volume occupied by long-range connections. It is assumed that the layout of cortical areas minimizes the total length of axons needed to join them [6]. Furthermore, severe metabolic constraints need to be satisfied by the cortex in order to achieve an energy efficient neural network design [7].

To explore the network architectures that satisfy the above mentioned constraints and properties, we use a coupled map lattice (CML). The latter were introduced by Kaneko, who also showed their importance in biology, particularly as models of neural networks [5]. Recently, Roerig et al. [10] determined the distribution of lateral connections in the primary visual cortex (V1) of ferrets. In contrast to often assumed Gaussian distributions, the probability to find two neurons connected in V1 shows a long tail, reminescent to an inverted power law. A CML with such a distribution was introduced by Raghavachari et al. [8] and named fractally coupled network. These authors showed that the degree to which such CML's synchronize depends upon the power law exponent. We have investigated the speed of information transfer in these networks and observed that fractally coupled networks are superior to nearest neighbor coupled, Gaussian coupled, and exponentially coupled systems [15]. It turned out that the fast information transfer can be inferred from the network topology requiring a fast decay of the probability distribution for short range connections, and a long tail for long range connections. As a consequence, we arrive at an optimized network topology by using a superposition of two power-law distributions: The first distribution accounts for the fast decay, and the second is responsible for the long tail. Indeed, as we will show in this paper, networks with this bi-power law architecture substantially increase the speed of information transfer. Moreover, synchronization in these networks takes place with a decreased number of connections per node and with a reduced total length of connections.

2. The Model

The architecture of the network is represented by the adjacency matrix A of a finite connected graph Γ with N nodes. According to the experimentally determined power law, we construct the adjacency matrix of the network by using that the probability that two nodes are connected

$$p_{i,j} = \theta \mid r_i - r_j \mid^{-\alpha} + (1-\theta) \mid r_i - r_j \mid^{-\beta} . \quad (1)$$

The exponents α and β determine the decay behavior of the distribution, r_i denotes the position on the lattice, and $\theta \in [0, 1]$ the partition coefficient. Using $\theta = 1$, the system can be changed from a globally coupled network ($\alpha = 0$) into a nearest neighbor coupled network ($\alpha \to \infty$). For $\alpha \to \infty, \beta = 0$ and $0 < \theta < 1$, the network is coupled to the nearest neighbor with probability 1, and to all other nodes with probability $(1 - \theta)$, down to the cutoff. Hence, we obtain a combined nearest neighbor- and random-coupled network. The former property warrants that the associated graph remains connected. For all intermediate values of α and β the network is fractally coupled. In the present study we are using a one dimensional system on a ring with reciprocal connections, implying that A is symmetric.

The dynamical system on Γ is a coupled map lattice with discrete time $t \in \mathbf{Z}$ and state variable x_i. The evolution of the system is described by the equations

$$x_i(t+1) = f(x_i(t)) + \varepsilon\left(\frac{1}{n_i}\sum_{j,j\sim i} f(x_j(t)) - f(x_i(t))\right) \quad (2)$$

for $i = 1 \ldots N$. In the rhs expression, f is a differentiable function mapping the interval I to itself. $\varepsilon \in [0, 1]$ is the coupling constant. n_i denotes the number of connections of node i, and $j \sim i$ indicates the presence of a connection between nodes i and j. Recently, Atay et al. [1] showed that the stability of a synchronized solution of Eqn. 2 can be established by the local condition

$$| e^{\mu}(1 - \varepsilon\lambda_k) |, \quad (3)$$

where μ is the Lyapunov exponent of f, and λ_k is the k^{th}-eigenvalue of the graph Laplacian Δ_Γ. The matrix L of Δ_Γ is defined as $L = I - \tilde{A}$, where I is the identity matrix and matrix $\tilde{A}$ is row-normalized. We used this criterion to determine whether a network with a given adjacency matrix synchronizes or not.

The information transfer in coupled map lattices can be studied by monitoring the propagation of a perturbation in these systems [3]. The flow of information is determined by two contributions. First, the chaotic instability of a map leads to an exponential growth of the initial perturbation, and second, the coupling results in a Gaussian spreading. To determine the speed v^* of the wave front, the convective Lyapunov exponent was introduced [4]. This then yields for the critical velocity

$$v^* = \sqrt{4D\mu}, \quad (4)$$

where D denotes the diffusion coefficient. We have shown that the spreading of the perturbation can be written in terms of a Markov process. This approach allows the determination of the diffusion coefficient and of the velocity without performing simulations. The explanation for this is that for these systems, time and space coordinates remain uncoupled during system evolution. Thus, D can be derived from a one dimensional random walk

$$D = \frac{(N)^2}{8\tau_{N/2}}, \quad (5)$$

if the system is perturbed at the central position. To determine $\tau_{N/2}$, two absorbing states at position $i = 0$ and $i = N + 1$ have to be introduced. The mean first passage time to reach the absorbing states can be calculated using the transition probability matrix P

$$\tau = (1 - P)^{-1}(1 \ldots 1)^T, \quad (6)$$

where P is given as $P = (1 - \varepsilon)I + \varepsilon\hat{N}$. Here, I denotes the identity matrix and $\hat{N}$ is equal to $\tilde{N}$, except that it accounts for the two absorbing states. Finally, $\tau_{N/2}$ is the entry of τ at position $N/2$.

3. Results

We first characterize the graph of a fractally coupled network. A typical probability distribution used in this study is displayed in Fig. 1A. The parameter values are $\alpha = 0.5$, $\beta = 10$ and $\theta = 0.3$ for a symmetrically connected network with $N = 512$ elements and a connection length cutoff at $M = 256$. This system is compared to a fractally coupled network with $\alpha = 0.7$ and $\theta = 1$ (Fig. 1B). Both systems are at the border that separates the parameter space into synchronous and non-synchronous behavior. The important point here is that the bi-power law distribution always remains below the one having one single exponent. The complete graph can be characterized by the eigenvalue spectrum of the adjacency matrix. For a random network, a semi circle is obtained, whereas a scale free network is characterized by a symmetric triangular shape. The spectral density of a fractally coupled network is asymmetric and reveals in the left part a peak-like structure (see Fig 1D). The origin of the peak might be due to the high probability for nearest neighbor coupling in these networks. It occurs in a pronounced fashion only if $\alpha > 1$. Although a similar asymmetry is observed in a bi-power law coupled network, the spectral density is already very close to that of a random-coupled network (Fig. 1C).

To follow the evolution of the CML, we used the logistic map

$$f(x) = 1 - ax^2 \quad (7)$$

to describe the dynamics of a single node (with $a = 1.9$). This map is chaotic, with a Lyapunov exponent of $\mu = 0.5419$. The coupling constant is set to $\varepsilon = n_i/(1 + n_i)$, which is close to 1 for all investigated networks. To estimate whether a network synchronizes or not (we consider complete synchronization only) we used Eqn. 3. The eigenvalues of the Laplacian L are semi positive. The smallest eigenvalue $\lambda_0 = 0$ corresponds to the constant eigenfunction. As shown in [1], it is sufficient to study eigenvalues larger than λ_0, where the one closest to zero $\lambda_1 > 0$ plays the dominant part. In all investigated networks, λ_1 described the transition from irregular to synchronized motion.

In order to study the transition to synchronization, we varied the partition parameter in the range $0 < \theta < 0.5$. It is

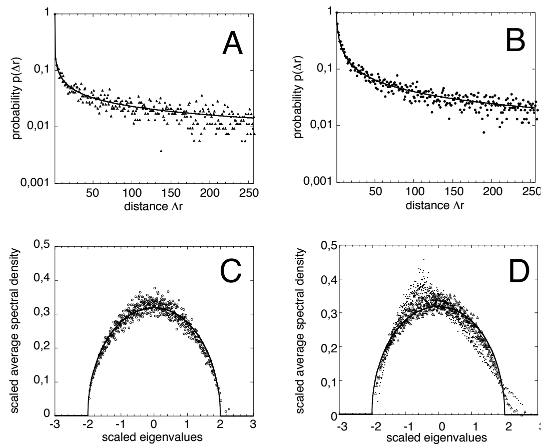

Figure 1: Characterization of the graph Γ of fractally coupled networks ($N = 512$). Probability distribution $p(\Delta r)$ for the presence of a connection between two nodes Δr apart. Full lines: calculated curves. Scatter plot: two realizations, full triangles for the example with $\alpha = 0.5, \beta = 10$ and $\theta = 0.23$ (A), dots for $\alpha = 0.7$ and $\theta = 1$ (B). (C), (D): Graph spectra corresponding to (A), (B), respectively. (D): Additional spectrum with $\alpha = 1.5$ and $\theta = 1$. The scaling factor is $\sqrt{np(1-p)}$, where p is the ratio of mean number of connections per node divided by $N - 1$. The full line represents the semi circle of a random-coupled network.

well known that for $\theta = 0$ and $\beta = 10$, the system does not synchronize, whereas for $\theta = 1$ and $\alpha = 0.5$, it does. Hence, by increasing θ, the range of coupling grows from nearest neighbors to system size. As shown in Fig. 2A, the transition takes place at $\theta = 0.21$, or at a mean node degree of $dg = 15$, see Fig. 2B. In contrast, a single power law distribution with $\alpha = 0.7$ yields an almost doubled mean degree of $dg = 29$.

For the energy consumption of the network, the mean number of connections per node is an important factor. Although this number is substantially reduced in bi-power law coupled networks, in purely random-coupled networks the average number of connections per node required for synchronization is even lower ($dg = 12$). However, as was stressed by Klyachko et al. [6], cortical neural networks appear to minimize their total length of connections. The calculations reveal for the single power law network and the bi-power law network a the total length of connections of 528876 and 309306, respectively. Interestingly, with 390087, the random-coupled network falls between these two numbers (where the results represent averages of 10 networks with $N = 512$ nodes each). Although the number of connections is reduced in random-coupled networks compared to a bi-power law connected system (12 versus 15 for N=512), the total length of connections is significantly larger in the former.

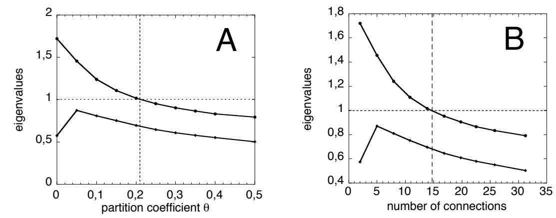

Figure 2: Transition to synchronization in fractally coupled map lattices (parameters: $\alpha = 0.5, \beta = 10$; average of 10 networks). Upper curve (circles) and lower curve (dimonds) represent the condition of Eqn. 3 for the second eigenvalue (A) Increasing θ from 0 to 0.5 reveals that the transition occurs at $\theta = 0.21$. This value corresponds to nodes with an average degree of 15 (B).

To investigate the speed of information transfer, we used Eqn. 4 and Eqn. 5. Since the Lyapunov exponent of a single map remains constant, in Fig. 3 only the square root of the diffusion coefficient is displayed, for the bi-power law (triangles), the single power law (squares) and the nearest-neighbor coupled network (circles). The abscissa represents the number of connections required to achieve the corresponding speed of information transfer. This is achieved by an increase of the cutoff length M, which limits the range of connections. Because these networks all are first-nearest-neighbor coupled, for $M = 1$ the curves start at the same point. The larger the cutoff length, the more is the curve affected by the probability distribution. For the bi-power law coupled network, the range of connections was varied from $M = 1$ to $M = 240$. Since the maximal value of M is 256, the flattening of the curve for large numbers of connections is a network-size effect. However, Fig. 3 clearly demonstrates that the velocity-enhancing effect of the bi-power law distribution persists over the full range.

4. Summary

According to the binding hypothesis, distributed information of the same object is aggregated by synchronization in the visual fields. Furthermore, it has been proposed that the cortical neural networks develop by minimizing the total length of connections. In addition, in a recent study of the lateral network in the primary visual cortex, an inverted power law distribution of connections was reported. To uncover the structure of a network that ideally fulfills these requirements, we explored networks of different architectures.

The analysis was performed using coupled map lattices.

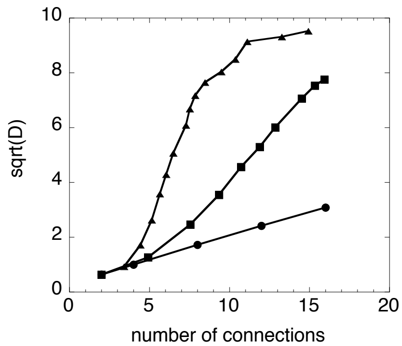

Figure 3: Information transfer in fractally coupled networks (Eqn. 4, Eqn. 5). The curves represent $\sqrt{D}$ versus the number of connections (average of 10 networks) for the bi-power law (triangles: $\alpha = 0.5, \beta = 2, \theta = 0.21$), the single power law (squares: $\alpha = 0.7, \theta = 1$) and nearest neighbor coupled networks (circles). Different number of connections were obtained by varying the cutoff length.

We constructed a connection scheme with a bi-power law distribution and compared it to a single power-law distribution. We first showed that the speed of information transfer can be significantly enhanced, and second, that the network synchronizes with a decreased number of connections (halved) and with a significantly reduced total length of connections ($\approx$ 60%). Finally, we showed that the total length remains markedly shorter than that of a purely random-coupled network.

Acknowledgments

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