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Decay of correlations of Soft-Aggregation of Processes

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Abstract— The problem of aggregating different stochastic process into a unique one that must be characterized based on the statistical knowledge of its components is a key point in the modeling of many complex phenomena such as the merging of traffic flows at network nodes.

Depending on the physical intuition on the interaction between the processes, many different aggregation policies can be devised, from averaging to taking the maximum in each time slot.

We here give a set of axioms defining a general aggregation operator and, based on some advanced results of functional analysis, we investigate how the decay of correlation of the original processes affect the decay of correlation (and thus possibly the self-similar features) of the aggregated process.

1. Introduction

This work is a generalization of [1] where the problem of characterizing the correlation profile of aggregated traffic fluxes is addressed.

In that work it is assumed that the traffic is slotted and that a slot can be either full or empty. Moreover, it is assumed that the merging between fluxes is performed by taking the $\max$ of the slot states.

We here consider a more general traffic model in which each slot can be partially filled with data. This immediately poses the problem of how data coming from different sources interact at a common node. It is not certainly sensible to assume that a simple $\max$ of the slot fill levels is the one and only possible model for such an interaction.

In fact, the simultaneous presence of data on more than one input port is likely to result in an output traffic that is larger than each of the incoming loads, up to the complete saturation of the slot capacity.

The key point of this paper is to leave the description of the aggregation mechanism free, specifying only that it must possess some common-sense properties listed in an axiomatic form. Starting from these axioms, and resorting to some recent results on the structure of this kind of connectives, we obtain that there is a one-to-one correspondence between the possible aggregations and the continuous non-decreasing real functions on the interval $[0, 1]$.

This intrinsic structure allow us to apply some elementary statistical analysis and derive that the covariance decay of the output flux is determined by the slowest covariance decay among the input processes, independently of the actual mechanism used to merge them.

2. Process and Aggregation Models

We will deal with discrete-time processes that, at each integer time steps k, associate a random variable x_k assuming values in $[0, 1]$.

This structure fits, for example, the need of modeling a slotted link whose slots can be filled by traffic units to different levels. An empty slot corresponding to a 0, a completely saturated slot corresponding to a 1.

When traffic fluxes come to a node they may be aggregated into a unique flux that can also be modeled by a slotted link.

Hence, the filling level of the incoming slots correspond to a filling level for the outcoming slot that can be computed by means of an aggregation. In the following we will assume that such an aggregation can be modeled by a so called s-norm [2] $\oplus : [0, 1]^2 \mapsto [0, 1]$ such that if $x_k^{(0)}, x_k^{(1)}, \ldots, x_k^{(m-1)}$ are the incoming fluxes then

$$y_k = x_k^{(0)} \oplus x_k^{(1)} \oplus \cdots \oplus x_k^{(m-1)} \tag{1}$$

is the filling level of the k-th outgoing slot.

The family of s-norm is formally defined as containing those functions such that

(s1) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$

(s2) $a \oplus b = b \oplus a$

(s3) $a \oplus 0 = a$

(s4) $a \geq c, b \geq d \Rightarrow a \oplus b \geq c \oplus d$

These axioms simply encode the intuitive need for a monotonic (s4) aggregation mechanism that is associative (s1) and commutative (s2) so that aggregations do not depend on the order in which we consider incoming fluxes, and such that void slot are unimportant (s3).

Some possible s-norms are the $\max$ operator $a \oplus b = \max\{a, b\}$, the probabilistic sum $a \oplus b = a + b - ab$, the saturating sum $a \oplus b = \min\{1, a + b\}$ and the operator

$$a \oplus_{\max} b = \begin{cases} \max\{a, b\} & \text{if } \min\{a, b\} = 0 \\ 1 & \text{otherwise} \end{cases}$$

From axioms we easily get that any s-norm satisfies $\max\{a, b\} \leq a \oplus b \leq a \oplus_{\max} b$ so that they actually span the range between the more "optimistic" view of the $\max$ (none but the heaviest incoming traffic flux affects the output) and the "pessimistic" behavior of $\oplus_{\max}$ (whenever all the aggregated fluxes are non-null the output is saturated to its maximum).

We will additionally require that the s-norm is continuous. With this, a straightforward generalization of the fundamental result in [3] allow us to say that there must be a possibly countable collection of open intervals $]a_i, b_i[$ and corresponding continuous nondecreasing functions $S_i : [a_i, b_i] \mapsto \mathbb{R}^+$ (the so called *generating functions* such that

$$a \oplus b = \begin{cases} S_i^*(S_i(a) + S_i(b)) & \text{if } a, b \in]a_i, b_i[\\ \max\{a, b\} & \text{otherwise} \end{cases}$$

where the "pseudo-inverse" S_i^* of S_i is defined as

$$S_i^*(\xi) = \begin{cases} a_i & \text{if } \xi \in [0, S_i(a_i)[\\ S_i^{-1}(\xi) & \text{if } \xi \in [S_i(a_i), S_i(b_i)] \\ b_i & \text{if } \xi \in [S_i(b_i), +\infty[\end{cases}$$

It is finally sensible to assume that, when $a, b > 0$ we also have $a \oplus b > \max\{a, b\}$ to indicate that, with the exception of a null contribution, no input can be actually neglected in the aggregation.

With this we get that the collection of intervals $[a_i, b_i]$ and associated functions S_i must actually consist of only one element. In fact, if there were at least two of them we may assume $0 \le a_1 < b_1 < a_2 < b_2 \le 1$ and compute $b_1 \oplus a_2 = \max\{b_1, a_2\} = a_2$ for the two numbers b_1 and a_2 that are strictly above zero. For analogous reasons we must have $a_1 = 0$ and $b_1 = 1$.

Hence, the family of s-norms in which we are interested is made of aggregation mechanisms that are univocally identified by a single continuous nondecreasing generating function $S : [0, 1] \mapsto \mathbb{R}^+$ by the definition

$$a \oplus b = S^*(S(a) + S(b))$$

Since only one generating function exists we may also assume $S(0) = 0$ to recast the definition of pseudo-inverse as

$$S^*(\xi) = \begin{cases} S^{-1}(\xi) & \text{if } \xi \in [0, S(1)] \\ 1 & \text{if } \xi \in]S(1), +\infty[\end{cases}$$

Note that, even if we have formally excluded $\oplus = \max$ from the class of s-norms we consider, we may approximate its behavior arbitrarily well considering, for example, $S(\xi) = \xi^p$ for $p \to \infty$. Note also that, for $p \to 0$, the same generator produces $\oplus_{\max}$. With this we know that the family of aggregation we analyze is able to span the whole range of optimistic-pessimistic choices allowed by the s-norm axioms.

Finally, the presence of a single generating function allows us to say that, in case of multiple aggregation, the output process can be written as

$$y_k = S^*\left(\sum_{i=0}^{m-1} S(x_k^{(i)})\right) \tag{2}$$

To prove this we may exploit the associativity of $\oplus$ and proceed by induction. For $m = 2$ the fact is intrinsic in the expression of $\oplus$ in terms of its generating function.

For $m > 2$ we may assume that the property holds for $m - 1$ and write

$$y_k = S^*\left(S\left(S^*\left(\sum_{i=0}^{m-2} S(x_k^{(i)})\right)\right) + S(x_k^{(m-1)})\right)$$

Two cases must be taken into account

- If $\sum_{i=0}^{m-2} S(x_k^{(i)}) \le S(1)$ then

$$S\left(S^*\left(\sum_{i=0}^{m-2} S(x_k^{(i)})\right)\right) = \sum_{i=0}^{m-2} S(x_k^{(i)})$$

  and (2) is proved.

- If $\sum_{i=0}^{m-2} S(x_k^{(i)}) > S(1)$ then

$$S\left(S^*\left(\sum_{i=0}^{m-2} S(x_k^{(i)})\right)\right) = S(1)$$

  and thus

$$y_k = S^*(S(1) + S(x^{(m-1)})) = 1$$

  that coincides with what (2) would give due to the non decreasing nature of S.

3. Auto-correlation trend and aggregation

Assuming to deal with processes x_k that are stationary, we consider the auto-covariance function

$$\begin{aligned} C_x(\tau) &= \mathbf{E}[x_0 x_\tau] - \mathbf{E}^2[x_0] \\ &= \int_{[0,1]^2} \alpha\beta \left[f_{x_0 x_\tau}(\alpha, \beta) - f_{x_0}(\alpha) f_{x_\tau}(\beta)\right] d\alpha d\beta \end{aligned}$$

where f_{x_0, x_τ} is the joint probability density of x_0 and x_τ while f_{x_0} and f_{x_τ} are the (identical) probability densities of x_0 and x_τ separately.

We aim at investigating how the asymptotic trends of the covariance functions $C_{x^{(i)}}(\tau)$ mix up to determine the asymptotic trend of the covariance function $C_y(\tau)$ of the aggregated process.

In particular we will show that the slowest decaying correlation dominates and determines the decay of correlation of the aggregation.

To do so, we decompose the proof into two parts. First we prove two Lemmas saying that

- the decay of correlation is dominated by the slowest trend when independent processes are summed
- the asymptotic relationship between any two decays of correlation does not change when applying a continuous nondecreasing transformation

To prove the first Lemma assume that m processes $w_k^{(i)}$ $(i = 0, 1, \ldots, m-1)$ are given and that $z_k = \sum_{i=0}^{m-1} w_k^{(i)}$.

Since the summands are independent we have

$$\begin{aligned} C_z(\tau) &= \mathbf{E}[z_0 z_\tau] - \mathbf{E}^2[z_0] = \\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} \mathbf{E}[w_0^{(i)} w_\tau^{(j)}] - \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} \mathbf{E}[w_0^{(i)} w_0^{(j)}] \\ &= \sum_{i=0}^{m-1} \mathbf{E}[w_0^{(i)} w_\tau^{(i)}] - \mathbf{E}^2[w_0^{(i)}] \\ &= \sum_{i=0}^{m-1} C_{w^{(i)}}(\tau) \end{aligned}$$

from which we easily get that if we may exclude exact cancellations, the slowest covariance decay sets the global covariance decay.

To address the second Lemma we may note that, considering two processes w_k and $z_k = F(w_k)$ where F is a continuous nondecreasing transformation we may define

$$g_\tau(\alpha, \beta) = f_{w_0, w_\tau}(\alpha, \beta) - f_{w_0}(\alpha) f_{w_\tau}(\beta)$$

to write

$$C_w(\tau) = \int_{[0,1]^2} \alpha\beta g_\tau(\alpha, \beta) d\alpha d\beta$$

and

$$C_z(\tau) = \int_{[0,1]^2} F(\alpha)F(\beta)g_\tau(\alpha,\beta)d\alpha d\beta$$

and realize that the vanishing of both covariance functions must depend on the vanishing trend of g as $\tau \to \infty$, i.e. on the mixing properties of the process [5].

We will assume that this vanishing trend is uniform from above and from below, i.e., that an infinitesimal function $\epsilon(\tau)$ and two constants $0 < A' \leq A''$ exist such that

$$A'\epsilon(\tau) \leq |g_\tau(\alpha,\beta)| \leq A''\epsilon(\tau)$$

for all points α, β with the possible exception of those belonging to a set whose probability vanishes when $\tau \to \infty$.

With this we may asymptotically write

$$A'\epsilon(\tau)\int_{[0,1]^2} \alpha\beta d\alpha d\beta \leq |C_w(\tau)| \leq A''\epsilon(\tau)\int_{[0,1]^2} \alpha\beta d\alpha d\beta$$

$$A'\epsilon(\tau)\int_{[0,1]^2} F(\alpha)F(\beta)d\alpha d\beta \leq |C_z(\tau)| \leq A''\epsilon(\tau)\int_{[0,1]^2} F(\alpha)F(\beta)d\alpha d\beta$$

confirming that the asymptotic trend of the two covariance functions is the same.

The assumption on the uniform vanishing of the mixing term g is somehow restrictive. It can be easily seen that it must hold for memory-one exact processes that assume only a finite number of values (in this case the $\epsilon(\tau)$ trend is exponential) (see e.g. [6]). It can be reasonably expected that, in the above setting, the memory-one property is not necessary to cause uniform mixing that has more to do with exactness and finiteness of the number of values (see, for example, [7] for a uniform mixing process that is self-similar).

Though a formal proof of this has not been perfected so far we will assume that the class of processes satisfying uniform mixing is large enough to comprise the interesting phenomena.

Once that these two Lemmas are available, we may resort to (2) to realize that the aggregated process is obtained by first passing all the incoming fluxes through the function S. This preserves the asymptotic behavior of their covariance functions. Then, these distorted versions of the inputs are then summed and the slowest trend dominates. The last application of the function S^* finally translates this dominant trend to the output stream.

4. Application to Second-order Self-Similarity

We may now recall the scenario entailing traffic fluxes to apply the above result and find that, self-similar traffic in one of the incoming fluxes is going to produce self-similar bursts in the output stream.

In fact, the most common definition of second-order self-similar traffic depends on the scale-invariant properties of the covariance function. In particular, given a process w_k its aggregated version of order m is defined as

$$w_k^{(m)} = \frac{1}{m}\sum_{j=0}^{m-1} w_{km+j}$$

and the m-th order covariance function as

$$C^{(m)}(\tau) = \mathbf{E}[w_0^{(m)}w_\tau^{(m)}] - \mathbf{E}^2[w_0^{(m)}]$$

The original process is said to be second-order asymptotically self-similar if [8] a number $\beta \in]0,1[$ exist such that

$$\frac{C^{(xA)}(0)}{C^{(A)}(0)} \sim x^{-\beta}$$

and

$$\frac{C^{(A)}(\tau)}{C^{(A)}(0)} \sim \tau^{-\beta}$$

when $A, \tau \to \infty$.

It is quite well known that the above conditions are met when $C^{(1)}(\tau) \sim \tau^{-\beta}$, when the covariance function has a polynomial decay.

With this, and recalling that the covariance slowest decay rate of the inputs is the one that dominates the covariance decay rate of aggregated traffic, one immediately derives that whenever an incoming flux has self-similar features they are transferred to the output.

Remark that this happens regardless of the particular aggregation mechanism, let it be as optimistic as any of the approximations of the $\max$ or more pessimistic.

5. Numerical Examples

We concentrate on processes generated by chaotic maps, namely by the simple fixed point double intermittency map defined as [9]

$$M(x) = \begin{cases} \dfrac{x}{\sqrt[m_1]{1-(2^{m_1}-1)x^{m_1}}} & \text{if } x \leq 1/2 \\[2ex] 1 - \dfrac{1-x}{\sqrt[m_2]{1-(2^{m_2}-1)(1-x)^{m_2}}} & \text{if } x > 1/2 \end{cases}$$

where $m_1 \geq 0$ and $m_2 \geq 0$ control the self-similarity behavior of the process through its sojourn times in a neighborhood of 0 and 1 respectively.

When $m_1, m_2 \to 0$ we obtain the classical Bernoulli shift whose decay of covariance is well known to be exponential $C(\tau) \sim 2^{-\tau}$. Then $m_1, m_2 > 0$ then we have polynomial covariance decay and thus self-similarity as previously defined.

To show the effect of s-norm aggregation we concentrate on the generator $S(\xi) = \xi^p$ that has already discussed. A graphic of $a \oplus b$ for $p = 0.5, 2, 10$ is reported in Figure 1. Note how $p = 10$ is very close to the $\max$ operator already considered in [1].

We then generate two processes $x_k^{(0)}$ and $x_k^{(0)}$, the first by means of a map M with $m_1 = m_2 = 0.1$ and the second with a Bernoulli shift. The two processes are aggregated by means of the s-norms corresponding to $p = 0.5$, $p = 2$ and $p = 10$ and covariances are estimated from a 10^6 samples chunk. Results are reported in Figure 2, 3, and 4.

Note how the different aggregation mechanisms actually produces different correlation trends at the output. Despite this, the righmost part of the aggregated trend is practically parallel to the self-similar trend revealing that self-similarity is inherited from the slowest decaying covariance.

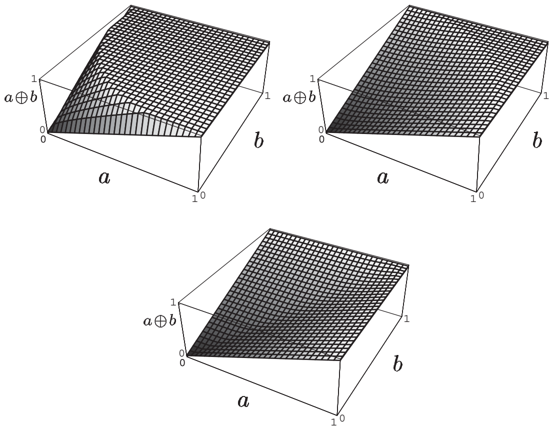

Figure 1: An example of s-norm corresponding to the generating functions $S(\xi) = \xi^{0.5}$, $S(\xi) = \xi^{2}$, $S(\xi) = \xi^{10}$.

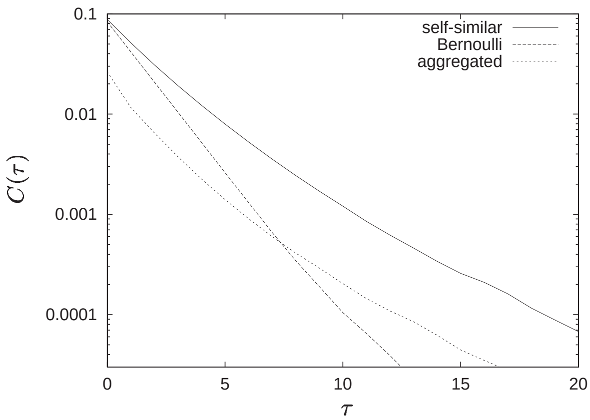

Figure 2: Aggregating a self-similar process and a Bernoullian process by menas of an s-norm with generating function $S(\xi) = \xi^{0.5}$.

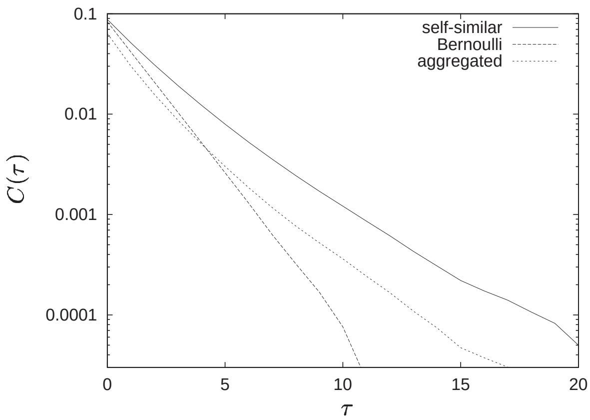

Figure 3: Aggregating a self-similar process and a Bernoullian process by menas of an s-norm with generating function $S(\xi) = \xi^{2}$.

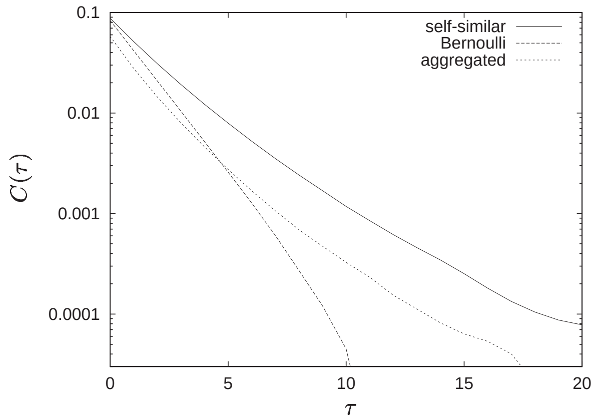

Figure 4: Aggregating a self-similar process and a Bernoullian process by menas of an s-norm with generating function $S(\xi) = \xi^{10}$.

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