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Lagrangian Formalism for Nonconservative Mechanical Systems with Nonholonomic Constraints

Hiroaki Yoshimura

†Department of Mechanical Engineering, Waseda University
3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan
‡Email: yoshimura@waseda.jp

Abstract—In the paper, we demonstrate a geometric approach to nonholonomic constrained systems with a nonconservative force field based on Lagrangian formalism. We first illustrate a Lagrangian system for a conservative mechanical system in the context of variational principle of Hamilton and then show that a mechanical system with nonholonomic constraints can be formulated on the tangent bundle of a configuration manifold by using Lagrange multipliers. We also investigate a nonconservative force field as a horizontal one-form on the tangent bundle and we finally demonstrate an intrinsic formulation of a nonconservative mechanical system with nonholonomic constraints by Lagrange-d'Alembert principle.

1. Introduction

Multibody systems such as robots, artificial satellites and vehicles are, in general, nonlinear mechanical systems with constraints. Specifically, dynamics of nonholonomic mechanical systems is no doubt a crucial problem in classical mechanics [3]. In analysis and design of such nonholonomic mechanical systems, geometric mechanics has played an essential role since early 90's [2]. The fundamental theory of geometric mechanics based on differential geometry was developed in late 60's in the field of mathematics [1], where the fundamentals of Lagrangian formalism on the tangent bundle as well as Hamiltonian formalism on the cotangent bundle associated with a configuration manifold have been common among researchers in the field. In the paper, we illustrate Lagrangian formalism for a nonconservative mechanical system with nonholonomic constraints specifically based on geometric mechanics by not only giving intrinsic expressions but also using local coordinates for applications. To do so, we first show the fundamental idea of Lagrangian formalism for a conservative mechanical system on the tangent bundle in the context of variational principle of Hamilton. Second we demonstrate the intrinsic formulation of a Lagrangian system by introducing Lagrangian forms and the Legendre transformation, where we primarily investigate the case of regular Lagrangians. Then, we study kinematical constraints given by a distribution on a configuration manifold, which will be considered as a distribution on the second tangent bundle and we also demonstrate a nonconservative force field associated with a horizontal one-form on the tangent bundle. Finally, we formulate dynamics of a nonconservative mechanical system with nonholonomic constraints by Lagrange-d'Alembert principle.

2. Variational Principle of Hamilton

2.1. Configuration Manifold and Tangent Bundle

First, we describe a geometric setting of Lagrangian formalism for a conservative mechanical system in the context of variational principle of Hamilton. Let us consider a conservative mechanical system whose configuration space is given by an n dimensional manifold Q and let q^i, $i = 1, ..., n$ be local coordinates for Q. Let $q(t),\ a \le t \le b$ be a motion of the mechanical system and hence generalized velocities are locally represented by $v = \sum_{i=1}^{n} v^i(\partial/\partial q^i)$, where $\partial/\partial q^1, ..., \partial/\partial q^n$ form a basis of the tangent space T_qQ at each point q. Let $TQ = \cup_{q\in Q} T_qQ$ be the tangent bundle of Q, which implies a *velocity phase space*. Let (q, v) be local coordinates for TQ. Furthermore, let T_q^*Q be the cotangent space at each point q and $T^*Q = \cup_{q\in Q} T_q^*Q$ be the cotangent bundle of Q, which denotes a *phase space*. Let (q, p) be local coordinates for T^*Q.

2.2. Euler-Lagrange Equations

In order to formulate a conservative mechanical system based on the Lagrangian formalism, we begin with the variational principle of Hamilton. Let Q be an n dimensional manifold and let L be a Lagrangian on TQ. For an arbitrary point $q \in Q$, the Lagrangian $L(q, v)$ can be regarded as a function on T_qQ. If the condition

$$\det\left[\frac{\partial^2 L}{\partial v^i \partial v^j}\right] \neq 0$$

is satisfied, then L is called *regular*. Define a path space joining two points q_1 and q_2 in Q with time interval $[a, b] \subset \mathbb{R}$ such that

$$\Omega(q_1, q_2, [a, b]) = \{q : [a, b] \to Q \mid q \text{ is a } C^2 \text{ curve}, \quad q(a) = q_1, q(b) = q_2\}.$$

Define an action functional $\mathfrak{S} : \Omega(q_1, q_2, [a, b]) \to \mathbb{R}$ by

$$\mathfrak{S}(q) = \int_a^b L\left(q(t), \dot{q}(t)\right) dt,$$

where $\dot{q} = dq/dt$. Then, we have the variational principle of Hamilton, which is given by the following theorem.

Theorem 2.1. *The curve $q_0 : [a, b] \to Q$ joining $q_1 = q_0(a)$ and $q_2 = q_0(b)$ satisfies the Euler-Lagrange equations*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{\partial L}{\partial q^i},$$

if and only if the action functional $\mathfrak{S} : \Omega(q_1, q_2, [a, b]) \to \mathbb{R}$ is critical at the point q_0, that is,

$$\delta \int_a^b L(q_0(t), \dot{q}_0(t))dt = 0.$$

The variation of the action functional $\mathfrak{S}$ is denoted by

$$\begin{aligned}\mathbf{d}\mathfrak{S}(q) \cdot v &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathfrak{S}(q_\epsilon) \\ &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_a^b L(q_\epsilon(t), \dot{q}_\epsilon(t))dt\end{aligned}$$

and hence we have

$$\mathbf{d}\mathfrak{S}(q) \cdot v = \int_a^b \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right) v^i dt + \left.\frac{\partial L}{\partial \dot{q}^i}\, v^i\right|_a^b.$$

Keeping the endpoints fixed, as the stationary condition $\mathbf{d}\mathfrak{S}(q) \cdot v = 0$ satisfies for all $v \in T_q\Omega(q_1, q_2, [a, b])$, we obtain the Euler-Lagrange equations

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = 0.$$

3. Lagrangian Forms on Tangent Bundle

3.1. Legendre Transformation

Let $L : TQ \to \mathbb{R}$ be a Lagrangian and recall the fiber derivative $\mathbb{F}L : TQ \to T^*Q$ is defined by

$$\mathbb{F}L(v) \cdot w = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} L(v + \epsilon\, w),$$

where $v, w \in T_qQ$, and $\mathbb{F}L(v) \cdot w$ is the derivative of L at v along the fiber T_qQ in the direction w. The map $\mathbb{F}L : TQ \to T^*Q$ is fiber-preserving and hence it maps the fiber T_qQ to the fiber T_q^*Q. The fiber derivative of L is locally represented by

$$\mathbb{F}L(q^i, v^i) = \left(q^i, \frac{\partial L}{\partial v^i}\right).$$

The map $\mathbb{F}L : TQ \to T^*Q$ is called the *Legendre transform* which is locally denoted by

$$p_i = \frac{\partial L}{\partial v^i}.$$

When the Legendre transform $\mathbb{F}L : TQ \to T^*Q$ is diffeomorphism, the Lagrangian L is said to be *hyperregular Lagrangian*.

3.2. Lagrangian One-Form and Two-Form

Given a hyperregular Lagrangian L. Since the cotangent bundle T^*Q of Q has naturally the canonical symplectic one-form Θ and two-form Ω. Then, by the diffeomorphism $\mathbb{F}L : TQ \to T^*Q$, we can define a one-form Θ_L and a two-form Ω_L on TQ such that

$$\Theta_L = (\mathbb{F}L)^*\Theta \quad \text{and} \quad \Omega_L = (\mathbb{F}L)^*\Omega,$$

each of which is called the Lagrangian one-form and the Lagrangian two-form. By the hyperregularity of the Lagrangian L, Ω_L is symplectic. Since $\Omega = -\mathbf{d}\Theta$ holds and $\mathbf{d}$ commutes with the pull-back, we have

$$\Omega_L = -\mathbf{d}\Theta_L.$$

The coordinate expression of the Lagrangian one-form Θ_L is given by

$$\Theta_L = \frac{\partial L}{\partial v^i} dq^i,$$

while the Lagrangian two-form Ω_L is locally denoted by

$$\Omega_L = \frac{\partial^2 L}{\partial v^i \partial q^j} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial v^i \partial v^j} dq^i \wedge dv^j.$$

The Lagrangian two-form Ω_L is represented by a skew-symmetric matrix such that

$$\Omega_L = \begin{bmatrix} \dfrac{\partial^2 L}{\partial v^i \partial q^j} - \dfrac{\partial^2 L}{\partial v^j \partial q^i} & \dfrac{\partial^2 L}{\partial v^i \partial v^j} \\ -\dfrac{\partial^2 L}{\partial v^i \partial v^j} & 0 \end{bmatrix}.$$

Since we consider the case that L is regular, $\partial^2 L/\partial v^i \partial v^j$ is nondegenerate and hence Ω_L is also nondegenerate.

4. Lagrangian Systems

4.1. Lagrangian Vector Field

Define a function $A : TQ \to \mathbb{R}$ called an action by

$$A(v) = \mathbb{F}L(v) \cdot v,$$

where $v \in T_qQ$. Further, define an energy by

$$E = A - L.$$

The coordinate expressions of the action A and the energy E are respectively represented by

$$A(q^i, v^i) = \frac{\partial L}{\partial v^i} v^i, \quad E(q^i, v^i) = \frac{\partial L}{\partial v^i} v^i - L(q^i, v^i).$$

Given a vector field X_E on TQ, if X_E satisfies the Lagrangian condition

$$\Omega_L(v)(X_E(v), w) = \mathbf{d}E(v) \cdot w$$

for all $v \in T_qQ$ and $w \in T_vTQ$, then X_E is said to be a *Lagrangian vector field* or a *Lagrangian system* for L.

Given a Lagrangian vector field $X_E : TQ \to TTQ$ and let $c(t) = (q(t), v(t))$, $t \in [a, b]$ be an integral curve of X_E. Then, the energy E is conserved such that

$$\begin{aligned}\frac{d}{dt}E(c(t)) &= \mathbf{d}E(c(t)) \cdot X_E(c(t)) \\ &= \Omega_L(c(t))\left(X_E(c(t)), X_E(c(t))\right) = 0,\end{aligned}$$

where $\dot{c}(t) = X_E(c(t))$ and we utilized the skew-symmetric property of Ω_L. Thus, we have an intrinsic expression of the Lagrangian system as

$$\mathbf{i}_{X_E}\,\Omega_L = \mathbf{d}E. \tag{1}$$

4.2. Second-Order Vector Field

Let us consider a submanifold of TTQ such that

$$T^{(2)}Q = \{w \in TTQ \mid T\tau_Q(w) = \tau_{TQ}(w)\},$$

where $\tau_Q : TQ \to Q$ is a canonical projection. In local coordinates, since $w = (q, v, \delta q, \delta v) \in TTQ$, an element of the submanifold $T^{(2)}Q$ satisfies the condition $v = \delta q$. So, if a vector field X_E on TQ satisfies $T\tau_Q \circ X_E = \mathrm{id}$, X_E is said to be a *second-order vector field*. In other words, a second-order vector field is defined as $X_E : TQ \to T^{(2)}Q$. Let $c(t)$ be an integral curve of X_E and let $(\tau_Q \circ c)(t)$ be an base integral curve of $c(t)$. The integral curve of X_E can be uniquely determined by the base integral curve $(\tau_Q \circ c)(t)$ with a given initial condition in TQ.

Theorem 4.1. *Let X_E be a Lagrangian vector field for $L : TQ \to \mathbb{R}$. Using local coordinates (q, v) for TQ, the integral curve $(q(t), v(t))$ of X_E satisfies the Euler-Lagrange equations*

$$\frac{dq^i}{dt} = v^i, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) = \frac{\partial L}{\partial q^i}, \quad i = 1, \ldots, n. \tag{2}$$

If the Lagrangian is regular, that is, Ω_L is nondegenerate, then X_E is to be second-order, and thus we have

$$\ddot{q}^j = M^{ij}\left(\frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i}\dot{q}^j\right), i, j = 1, \ldots, n, \tag{3}$$

where

$$[M^{ij}] = \left[\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right]^{-1}.$$

Notice that $q^i(t)$ is a base integral curve of X_E.

5. Constraints and Virtual Work Principle

5.1. Constraint Distributions

Kinematical constraints are represented by a constraint distribution $D \subset TQ$ on a configuration manifold Q. The distribution D is defined by, at each $q \in Q$,

$$D_q = \{v_q \in T_qQ \mid \langle \omega^r(q), v_q\rangle = 0 \text{ for all } \omega^r(q)\}, \tag{4}$$

where $\omega^r, r = 1, \ldots, m$ are independent one-forms locally denoted by

$$\omega^r(q) = \sum_{i=1}^{n} a_i^r(q)\, dq^i,\ r = 1, \cdots, m;\ m < n.$$

If any vector field X and Y on Q satisfies $[X, Y] \in D$, then D is completely integrable and constraints are said to be holonomic, otherwise nonholonomic. The motion of a mechanical system is said to be constrained if $\dot{q}(t) \in D_{q(t)}$ satisfies for each time $a \le t \le b$.

Using the map $\tau_Q : TQ \to Q$, the distribution $\widetilde{D}$ on TQ is defined by

$$\widetilde{D} = (T\tau_Q)^{-1}(D) \subset TTQ,$$

which is locally described by

$$\widetilde{D} = \{V \in TTQ \mid \langle(\tau_Q)^*\omega^r, V\rangle = 0\}.$$

Then, restricting $\widetilde{D}$ to $D \subset TQ$, one can obtain

$$\Delta = \widetilde{D} \cap TD \subset T_DTQ.$$

Note that an annihilator Δ° of Δ is defined by

$$\Delta^\circ = \{\alpha \in T_D^*TQ \mid \langle\alpha, w\rangle = 0 \text{ for all } w \in \Delta\},$$

where we note that $(\tau_Q)^*\omega^r$, $r = 1, ..., m$ form a basis of the annihilator Δ° such that $\alpha = \sum_{r=1}^{m}\mu_r(\tau_Q)^*\omega^r$.

5.2. Force Field and Virtual Work Principle

Let $\tau_Q : TQ \to Q$ be a canonical projection. A force field is a fiber-preserving map $F : TQ \to T^*Q$ over the identity, which induces a horizontal one-form $(\tau_Q)^*F$ on TQ such that

$$(\tau_Q)^*F(q, v) \cdot w = \langle F(q, v), T\tau_Q(w)\rangle,$$

where $(q, v) \in TQ$ and $w \in T_{(q,v)}TQ$. In local coordinates, a horizontal one-form $(\tau_Q)^*F$ is denoted by $(\tau_Q)^*F = (q, v, F, 0)$ and $w \in T_{(q,v)}TQ$ is given by $w = (q, v, \delta q, \delta v)$. So, we have $T\tau_Q(w) = (q, \delta q)$. Hence, the force field F is locally described by

$$F(q, v) = \sum_{i=1}^{n} F_i(q, v)\, dx^i.$$

If kinematical constraints are imposed on mechanical systems, whether holonomic or nonholonomic, we need to consider a constraint force field. The virtual work principle asserts that a constraint force F^c associated with the constraint distribution in equation (4) takes its value in D_q° for each $q \in Q$, which is represented such that

$$\langle F^c, \delta q\rangle = (\tau_Q)^*F^c \cdot w = 0,$$

where $\delta q = T\tau_Q(w) \in D_q$, $F^c \in D_q^\circ$ and $w \in \Delta$. Hence, we have

$$(\tau_Q)^*F^c = \sum_{r=1}^{m}\mu_r(\tau_Q)^*\omega^r \in \Delta^\circ. \tag{5}$$

6. Lagrange-d'Alembert Principle

6.1. Equations of Motion for Nonholonomic Systems

The Lagrange-d'Alembert Principle describes that equations of motion can be formulated by

$$\delta \int_a^b L(q,\dot{q})\,dt = 0, \tag{6}$$

where variations $\delta q(t)$ of the curve $q(t)$ is so chosen that $\delta q(t) \in D_{q(t)}$ for each $t \in [a,b]$ with $\delta q(a) = \delta q(b) = 0$. In the above, we take variations $\delta q(t)$ before imposing the constraints. In other words, the constraints are not imposed on the family of curves defining the variation, but the variations are chosen such that $\delta q(t) \in D_{q(t)}$ satisfies. The Lagrange-d'Alembert principle in equation (6) becomes

$$\left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} \right) \delta q^i = 0, \tag{7}$$

where $\delta q(t)$ is satisfied $\delta q(t) \in D_{q(t)}$ for each $t \in [a,b]$. Thus, we obtain the Lagrange-d'Alembert equations of motion as

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} + \sum_{r=1}^{m} \mu_r a_i^r = 0,\ i = 1, \cdots, n. \tag{8}$$

Note that the Lagrange-d'Alembert equations accompany with m kinematical constraints:

$$\sum_{i=1}^{n} a_i^r \, dq^i = 0,\ r = 1, \cdots, m;\ m < n.$$

Recall the intrinsic expression of Euler-Lagrange equations is denoted by equation (1) and hence the intrinsic expression of Lagrange-d'Alembert principle in equation (7) is given by

$$(\mathbf{i}_{X_E}\,\Omega_L - \mathbf{d}E) \cdot w = 0,$$

where $w \in \Delta$. Then, the intrinsic expression of the Lagrange-d'Alembert equations (8) is to be

$$\mathbf{i}_{X_E}\,\Omega_L - \mathbf{d}E = \sum_{r=1}^{m} \mu_r (\tau_Q)^* \omega^r.$$

6.2. Mechanical Systems with External Forces

Let $F^e : TQ \to T^*Q$ be a nonconservative external force field. The integral Lagrange-d'Alembert principle is represented by

$$\delta \int_a^b L(q,\dot{q})\,dt + \int_a^b F^e \,\delta q \,dt = 0, \tag{9}$$

where we choose variations $\delta q(t)$ of the curve $q(t)$ such that $\delta q(t) \in D_{q(t)}$. Keeping the endpoints fixed, the first term of the left-hand side of equation (9) is transformed into

$$\delta \int_a^b L(q,\dot{q})\,dt = \int_a^b \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} \right) \delta q^i.$$

Thus, we derive the Lagrange-d'Alembert equations for mechanical systems with an external force by employing Lagrange multipliers μ_r such that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \sum_{r=1}^{m} \mu_r a_i^r + F_i^e,\ i = 1, \cdots, n,$$

which combines with m kinematical constraints. The external force field $F^e : TQ \to T^*Q$ induces a horizontal one-form on TQ as

$$(\tau_Q)^* F^e(q,v) \cdot w = \left\langle F^e(q,v), T\tau_Q(w) \right\rangle,$$

where $(q,v) \in D \subset TQ$ and $w \in \Delta \subset T_{(q,v)}TQ$, and hence the intrinsic expression of the Lagrange-d'Alembert equations of motion is denoted by

$$\mathbf{i}_{X_E}\,\Omega_L - \mathbf{d}E = \sum_{r=1}^{m} \mu_r (\tau_Q)^* \omega^r + \mu_r (\tau_Q)^* F^e.$$

In the above, equations of motion together with m kinematical constraints consist of a complete set of system equations for a nonholonomic mechanical system with a nonconservative external force field.

7. Conclusions

We demonstrated a geometric approach to mechanical systems with nonholonomic constraints based on Lagrangian formalism in the context of variational principles. We first formulated Euler-Lagrange equations for a conservative system with no constraints and then investigated Lagrange-d'Alembert equations of motion for a conservative system with nonholonomic constraints. Last we considered a nonholonomic mechanical system with a nonconservative force on the tangent bundle.

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