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Singular value analysis of nonlinear operators: Application to Hankel operators

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Abstract—This paper presents a novel characterization of singular values of nonlinear operators. Although eigenvalue and spectrum analysis for nonlinear operators has been studied by many researchers in mathematics literature, singular value analysis has not been investigated so much. In this paper, a novel framework of singular value analysis is proposed which is closely related to the operator gain. The proposed singular value analysis is based on the eigenvalue analysis of a special class of nonlinear operators called differentially self-adjoint. Some properties of those operators are clarified which are natural generalization of the linear case results. A sufficient condition for the existence of singular values is provided. Furthermore, application of the proposed method to singular value analysis of nonlinear Hankel operators, which play important roles in nonlinear balanced realization and model reduction, demonstrates its effectiveness. The proposed analysis tools are expected to play an important role in nonlinear control systems theory as in the linear case.

1. Introduction

Eigenvalue analysis with the related techniques is one of the most beneficial tools in many scientific research fields. In particular, eigenvalue and singular value analysis plays a crucial role in linear control systems theory. It is quite natural to consider how to generalize these tools for nonlinear operators, whereas they are originally used for linear operators. In fact, there are several papers on eigenvalue and spectrum analysis for *nonlinear* operators in mathematics literature [3, 12, 10, 1].

Let us consider a Banach space X with a field $\mathbb{K}$ and a linear operator $A : X \to X$. Its *eigenvalue* λ and the corresponding *eigenvector* x are obtained by solving

$$Ax = \lambda x, \quad \lambda \in \mathbb{K}, \quad x(\neq 0) \in X.$$

Here λ rendering $A - \lambda I$ non-invertible is called a *spectrum* of A. The nonlinear version of this eigenvalue problem is formulated in a similar way as follows. Consider a nonlinear operator $f : X_0 \to X$ with $X_0 \subset X$. Its *eigenvalue* λ and the corresponding *eigenvector* x are obtained by solving

$$f(x) = \lambda x, \quad \lambda \in \mathbb{K}, \quad x(\neq 0) \in X.$$

Here λ rendering $f - \lambda I$ non-invertible is called a *spectrum* of f. The above nonlinear eigenvalue problem is a natural generalization of the linear case.

On the other hand, nonlinear versions of singular value problems were not investigated so much. This is because the definition of a nonlinear version of *adjoint* operators are not clear. In the linear case, singular vectors x's of a linear operator A are characterized by the eigenvectors of A^*A with A^* the adjoint of A, and the corresponding singular values are given by square roots of the eigenvalues of A^*A. See e.g. [14]. Although there are some research on adjoints of nonlinear operators [2, 4, 13, 9], its direct application does not derive any framework for singular value analysis so profitable as that in the linear case.

The objective of this paper is to provide a natural definition of singular values of nonlinear operators and to clarify some of their properties. First of all, recall that singular values in the linear case has a close relationship to the operator gain. A new definition of singular values of nonlinear operators is proposed based on their gain analysis. Then it is shown that thus defined singular values can be calculated by solving a special class of nonlinear eigenvalue problems with respect to a *differentially self-adjoint* operator which is a nonlinear counterpart of a self-adjoint operator in the linear case. Some of their properties related to the existence of singular values are clarified. Furthermore, the proposed method is applied to singular value analysis of nonlinear Hankel operators which play important roles in nonlinear balanced realization and model reduction.

In this paper, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ denotes a field where $\mathbb{R}$ and $\mathbb{C}$ denote the space of real numbers and that of complex numbers, respectively. The symbol $\mathbb{N}$ denotes the space of natural numbers. The operators $\mathrm{d}(\cdot)$ and $\mathbf{d}(\cdot)$ denote Fréchet derivative (for conventional operators) and exterior derivative (for differential forms), respectively, and the word 'differentiable' stands for 'Fréchet differentiable'. The symbols $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ with a complex number $x \in \mathbb{C}$ denote its real part and the imaginary part, respectively. The product $\langle \cdot, \cdot \rangle$ denotes the inner product for the corresponding Hilbert space with the field $\mathbb{K}$. The symbols S_r and D_r denote a sphere $S_r(X) := \{x \in X \mid \|x\| = r\}$ and a disk $D_r(X) := \{x \in X \mid \|x\| \leq r\}$, respectively. The symbol $\mathrm{T}_x M$ denotes the tangent space of M at x. All proofs are omitted for the reason of space. Please contact the author to obtain them.

2. Singular values of nonlinear operators

First of all, recall the definition of singular values in the linear case in order to show the line of thinking in the nonlinear case. In the linear case, singular values and singular vectors of a linear (compact) operator $A : X \to Y$ with Hilbert spaces X and Y are characterized by the eigenvalue problem of $A^*A : X \to X$

$$A^*Ax = \lambda x, \quad \lambda \in \mathbb{K}, \ x(\neq 0) \in X. \tag{1}$$

Here the solution x is called a (right) *singular vector* of A. The eigenvalue λ is always *real* and non-negative because A^*A is *self-adjoint* and positive semi-definite. So the corresponding *singular value* can be defined by

$$\sigma = \sqrt{\lambda} \ \left(= \frac{\|Ax\|}{\|x\|} \right). \tag{2}$$

Singular values are important because it characterizes the operator gain by

$$\|A\| := \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \frac{\|Ax\|}{\|x\|} = \sup \sigma. \tag{3}$$

How can we define singular values for nonlinear operators? Recall also that, when x is a singular vector corresponding to a nonzero singular value, then it is a critical point of the square of the input-output ratio $\|Ax\|^2/\|x\|^2$ under the constraint $x \in S_1(X)$ [14], that is,

$$\mathrm{d}\left(\frac{\|Ax\|^2}{\|x\|^2} \right)(\mathrm{d}x) = 0, \quad \forall \mathrm{d}x \ \text{ s.t. } (x, \mathrm{d}x) \in \mathrm{T}_x S_1(X).$$

Here we adopt this relationship as the starting point to define singular vectors of nonlinear operators. Consider a nonlinear operator $g : X_0 \to Y$ with an open set $X_0 \subset X$ containing 0. Take an arbitrary positive constant $r > 0$ and consider a similar problem finding critical points of the square of the input-output ratio under a constraint $x \in S_r(X) \cap X_0$.

$$\begin{aligned} &\mathrm{d}\left(\frac{\|g(x)\|^2}{\|x\|^2} \right)(\mathrm{d}x) = 0 \\ &\forall \mathrm{d}x \ \text{ s.t. } (x, \mathrm{d}x) \in \mathrm{T}_x(S_r(X) \cap X_0). \end{aligned} \tag{4}$$

Here an additional parameter r is introduced because the input-output ratio of a nonlinear operator varies according to the input magnitude r, differently from the linear case.

In the nonlinear case, the singular vectors satisfying (4) is characterized by the following nonlinear eigenvalue problem.

Theorem 1 *Consider Hilbert spaces X and Y with a field $\mathbb{K}$, and a bounded nonlinear operator $g : X_0 \to Y$ with an open set X_0 satisfying $0 \in X_0 \subset X$. Suppose that g is differentiable. Then x is a solution of (4) if and only if it satisfies*

$$(\mathrm{d}g(x))^* g(x) = \lambda\, x, \quad \lambda \in \mathbb{R}, \ x(\neq 0) \in X_0. \tag{5}$$

This property motivates us to characterize singular values and singular vectors of nonlinear operators as follows.

Definition 1 Consider Hilbert spaces X and Y with a field $\mathbb{K}$, and a differentiable bounded nonlinear operator $g : X_0 \to Y$ with an open set X_0 satisfying $0 \in X_0 \subset X$. An eigenvector of the operator $x \mapsto (\mathrm{d}g(x))^* g(x)$ corresponding to a *real* eigenvalue, that is, $x \in X_0$ satisfying (5) is called a *singular vector* of g and the corresponding input-output ratio defined by

$$\sigma = \frac{\|g(x)\|}{\|x\|}$$

with the singular vector x is called a *singular value* of g.

Note that Equation (5) is a natural nonlinear generalization of the singular value problem in the linear case (1). The reason why we adopt the second equation of (2) as the definition of singular values of nonlinear operators instead of the first one, is because λ in (5) can be negative. Furthermore, this definition yields the property

$$\|g\| := \sup_{x \in X_0} \frac{\|g(x)\|}{\|x\|} = \sup \sigma$$

as in the linear case (3), because the input maximizing the input-output ratio $\arg\sup(\|g(x)\|/\|x\|)$ has to satisfy Equation (4). Namely, nonlinear singular values are also closely related to the operator gain.

Remark 1 The author has provided a similar definition of singular values for nonlinear Hankel operators in [5]. This definition works quite nicely for Hankel operators, and nonlinear balanced realization and model reduction procedure are obtained consequently [6, 7]. Another important example of the proposed singular values can be found in L_2 gain analysis [15]. In fact, investigating the singular values for L_2 stable nonlinear input-output systems is equivalent to analyzing the solution of the corresponding Hamiltonian extension giving the solution of L_2 gain analysis of the original operator [7].

3. Differentially self-adjoint operators

In order to investigate the solution of (5), we need to characterize a nonlinear version of a *self-adjoint* operator, since the eigenstructure of such operators play an important role in investigating singular values of linear operators. Let us define *differentially self-adjoint* operators as follows.

Definition 2 Consider a Hilbert space X with a field $\mathbb{K}$, and a bounded nonlinear operator $f : X_0 \to X$ with an open set X_0 satisfying $0 \in X_0 \subset X$. The operator f is said to be *differentially self-adjoint* if it is differentiable and if $\mathrm{d}f(x) : X \to X$ is self-adjoint for all $x \in X_0$.

An intuitive motivation of this definition is explained by the following lemma and corollary.

Lemma 1 *Consider a Hilbert space X with a field $\mathbb{K}$, and a bounded nonlinear operator $h : X_0 \to \mathbb{R}$ with an open set X_0 satisfying $0 \in X_0 \subset X$. Suppose that h is continuously differentiable and that there there exists an operator $f : X_0 \to X$ satisfying*

$$\mathrm{d}h(x)(\mathrm{d}x) = \mathrm{Re}\langle f(x), \mathrm{d}x\rangle. \tag{6}$$

Then the operator f is differentially self-adjoint.

Corollary 1 *Consider Hilbert spaces X and Y with a field $\mathbb{K}$, and a bounded nonlinear operator $g : X_0 \to Y$ with an open set X_0 satisfying $0 \in X_0 \subset X$. Suppose that g is continuously differentiable. Then the operator $f : X_0 \to X$ defined by*

$$f(x) := (\mathrm{d}g(x))^* g(x) \tag{7}$$

is differentially self-adjoint.

Therefore, any singular value problem reduces down to an eigenvalue problem with respect to a special class of operators called differentially self-adjoint.

The final objective of this section is to provide a converse result of of Lemma 1. To this end, let us state the following lemma.

Lemma 2 *Consider a Hilbert space X with a field $\mathbb{K}$, and a bounded nonlinear operator $f : X_0 \to X$ with any simply connected open set X_0 satisfying $0 \in X_0 \subset X$. Suppose that f is differentially self-adjoint. Then*

$$\langle f(x), x\rangle \in \mathbb{R}, \quad \forall x \in X_0.$$

Using this lemma, we can prove a converse result of Lemma 1, which is a variation of Stokes's theorem.

Theorem 2 *Consider a Hilbert space X with a field $\mathbb{K}$, and a bounded nonlinear operator $f : X_0 \to X$ with a simply connected open set X_0 satisfying $0 \in X_0 \subset X$. Suppose that f is differentiable. Then f is differentially self-adjoint if and only if there exists an operator $h : X_0 \to \mathbb{R}$ satisfying*

$$\mathrm{d}h(x)(\mathrm{d}x) = \mathrm{Re}\langle f(x), \mathrm{d}x\rangle. \tag{8}$$

4. Eigenvalue analysis of differentially self-adjoint operators

This section investigates the solution structure of the eigenvalue problems of differentially self-adjoint operators based on the results derived in the previous sections, which is particularly useful for singular value analysis of nonlinear operators as explained in Theorem 1.

Theorem 3 *Consider a Hilbert space X with a field $\mathbb{K}$, and a bounded nonlinear operator $f : X_0 \to X$ with a simply connected open set X_0 satisfying $0 \in X_0 \subset X$. Suppose that f is differentially self-adjoint. Then all eigenvalues of f are real. Furthermore, if f satisfies (7), then all eigenvectors of f are singular vectors of g.*

This theorem allows us to concentrate on *real* eigenvalues when we treat differentially self-adjoint nonlinear operators, as in the linear case.

The final objective of this paper is to obtain a some conditions on the existence of singular values of nonlinear operators (i.e., eigenvectors of differentially self-adjoint operators).

Theorem 4 *Consider the Hilbert space $X = \mathbb{R}^n$ with the field $\mathbb{K} = \mathbb{R}$, and a bounded nonlinear operator $f : X_0 \to X$ with a simply connected open set X_0 satisfying $0 \in X_0 \subset X$. Suppose that f is differentially self-adjoint, and that $\mathrm{d}f(0) : X \to X$ has n distinct eigenvalues. Then there exists a set $D_r(\mathbb{R}) \subset \mathbb{R}$ satisfying $\{x \mid \|x\| \in D_r(\mathbb{R})\} \subset X_0$, and a set of differentiable operators $\lambda_i : D_r(\mathbb{R}) \to \mathbb{R}$'s and $x_i : D_r(\mathbb{R}) \to X_0$'s satisfying*

$$f(x_i(s)) = \lambda_i(s)\, x_i(s), \quad \|x_i(s)\| = |s|, \ s \in D_r(\mathbb{R}). \tag{9}$$

Furthermore, we can prove a relationship between the eigenvalues of f and the singular values of g. Recall the linear case and let σ_i's and λ_i's denote the singular values of $A : X \to Y$ and the eigenvalues of $A^*A : X \to X$. Then clearly we have

$$\lambda_i = \sigma_i^2 \tag{10}$$

due to the definition (2). The nonlinear counterpart of this equation is given by the following theorem.

Theorem 5 *Consider Hilbert spaces $X = \mathbb{R}^n$ and Y with a field $\mathbb{K}$, and a bounded nonlinear operator $g : X_0 \to Y$ with an open set X_0 satisfying $0 \in X_0 \subset X$. Suppose that the operator f defined by (7) satisfies the assumptions in Theorem 4. Then the singular values $\sigma_i(s)$'s of g defined by $\sigma_i(s) := \|g(x_i(s))\|/\|x_i(s)\|$ and the eigenvalues $\lambda_i(s)$'s of f satisfy*

$$\lambda_i(s) = \sigma_i(s)^2 + s\,\sigma_i(s)\frac{\mathrm{d}\sigma_i(s)}{\mathrm{d}s}. \tag{11}$$

Further, the converse relation is given by

$$\sigma_i(s)^2 = \frac{2}{s^2}\int_0^s s\,\lambda_i(s)\mathrm{d}s. \tag{12}$$

Theorem 5 shows the fact that there is a one-to-one relationship between $\lambda_i(s)$ and $\sigma_i(s)$. In the linear case, both $\sigma_i(s)$'s and $\lambda_i(s)$'s are constant, so Equations (11) and (12) recover the straightforward relationship (10).

5. Application to Hankel singular value analysis

This section explains how the singular value analysis framework developed in the previous sections apply to nonlinear Hankel operators which play important roles in nonlinear balanced realization and model reduction.

5.1. Hankel operators

Let us consider a nonlinear *Hankel operator* $\mathcal{H} : U \to Y$ defined on Hilbert spaces U and Y. Here we suppose that $\mathcal{H}$ can be decomposed as

$$\mathcal{H} = O \circ C \tag{13}$$

with the *controllability operator* $C : U \to X$ and the *observability operator* $O : X \to Y$ where C is surjective and X is also a Hilbert space. Typical examples of $\mathcal{H}$ are related to the following dynamical systems. See [8] for the details.

Example 1 Consider an asymptotically stable finite dimensional continuous-time linear system

$$\left\{\begin{array}{rcl} \dot{x} &=& Ax + Bu \\ y &=& Cx + Du \end{array}\right. .$$

It's controllability operator $C : L_2^m(\mathbb{R}_+) \to \mathbb{R}^n$ and observability operator $O : \mathbb{R}^n \to L_2^p(\mathbb{R}_+)$ are defined by

$$\begin{array}{rcl} x^0 &=& C(u) := \displaystyle\int_0^\infty \mathrm{e}^{A\tau} Bu(\tau)\mathrm{d}\tau \\ y &=& O(x^0) := C\mathrm{e}^{At}x^0. \end{array}$$

Its Hankel operator is given by the composition (13) with $U = L_2^m(\mathbb{R}_+)$, $X = \mathbb{R}^n$ and $Y = L_2^p(\mathbb{R}_+)$.

Example 2 Consider an L_2-stable finite dimensional continuous-time nonlinear system

$$\left\{\begin{array}{rcl} \dot{x} &=& f(x, u, t) \\ y &=& h(x, u, t) \end{array}\right. .$$

It's controllability operator $C : L_2^m(\mathbb{R}_+) \to \mathbb{R}^n$ and observability operator $O : \mathbb{R}^n \to L_2^p(\mathbb{R}_+)$ are defined by

$$x^0 = C(u) \quad : \quad \left\{\begin{array}{rcl} \dot{x} &=& -f(x, u, t) \quad x(\infty) = 0 \\ x^0 &=& x(0) \end{array}\right. \tag{14}$$

$$y = O(x^0) \quad : \quad \left\{\begin{array}{rcl} \dot{x}(t) &=& f(x, 0, t) \quad x(0) = x^0 \\ y &=& h(x, 0, t) \end{array}\right. \tag{15}$$

Its Hankel operator is given by the composition (13) with $U = L_2^m(\mathbb{R}_+)$, $X = \mathbb{R}^n$ and $Y = L_2^p(\mathbb{R}_+)$.

Example 3 Consider an ℓ_2-stable finite dimensional discrete-time nonlinear system

$$\left\{\begin{array}{rcl} x(t+1) &=& f(x(t), u(t), t) \\ y(t) &=& h(x(t), u(t), t) \end{array}\right. .$$

It's controllability operator $C : \ell_2^m(\mathbb{Z}_+) \to \mathbb{R}^n$ and observability operator $O : \mathbb{R}^n \to \ell_2^p(\mathbb{Z}_+)$ are defined by

$$x^0 = C(u) : \left\{\begin{array}{rcl} x(t-1) &=& f(x(t), u(t), t) \quad x(\infty) = 0 \\ x^0 &=& x(0) \end{array}\right.$$

$$y = O(x^0) : \left\{\begin{array}{rcl} x(t+1) &=& f(x(t), 0, t) \quad x(0) = x^0 \\ y(t) &=& h(x(t), 0, t) \end{array}\right. .$$

Its Hankel operator is given by the composition (13) with $U = L_2^m(\mathbb{Z}_+)$, $X = \mathbb{R}^n$ and $Y = L_2^p(\mathbb{Z}_+)$.

Here we investigate the singular value structure of Hankel operators which is a generalized version of the result in [5]. This investigation will derive a balancing and model reduction procedure which are applicable to a much wider class of nonlinear systems such as time-varying systems, input-non-affine systems, discrete-time systems.

The controllability and observability functions $L_c : X \to \mathbb{R}_+$ and $L_o : X \to \mathbb{R}_+$ with respect to the Hankel operator $\mathcal{H}$ given in (13) are defined by

$$L_c(x^0) := \inf_{\substack{C(u)=x^0 \\ u \in U}} \frac{1}{2}\|u\|^2 \tag{16}$$

$$L_o(x^0) := \frac{1}{2}\|O(x^0)\|^2. \tag{17}$$

If the pseudo-inverse $C^\dagger : X \to U$ of $C : U \to X$ defined by

$$C^\dagger(x^0) := \arg \inf_{\substack{C(u)=x^0 \\ u \in U}} \|u\| \tag{18}$$

exists, then L_c can be written as

$$L_c(x^0) = \frac{1}{2}\|C^\dagger(x^0)\|^2.$$

5.2. Singular value analysis of Hankel operators

Application of Theorem 1 to the nonlinear Hankel operator given in (13) yields the following corollary.

Corollary 2 *Suppose the Hankel operator $\mathcal{H} : U \to Y$ is Fréchet differentiable. Then a $v \in U$ is a singular vector of $\mathcal{H}$ if and only if*

$$(\mathrm{d}\mathcal{H}(v))^* \circ \mathcal{H}(v) = \lambda\, v, \quad \lambda \in \mathbb{R}, \quad v (\neq 0) \in U. \tag{19}$$

Here Equation (19) characterizes all stationary (critical) inputs. For the objective of Hankel theory in control, we are only interested in such stationary inputs in the image space of $C^\dagger$, see [5] for a detailed discussion on this matter. Therefore what we have to solve here is Equation (19) and

$$v \in \operatorname{Im} C^\dagger. \tag{20}$$

We call investigation of the solution of the above equation "singular value analysis of $\mathcal{H}$". Here the solution v is "a singular vector" and the corresponding scalar σ defined by

$$\sigma = \frac{\|\mathcal{H}(v)\|}{\|v\|} \tag{21}$$

is called a "singular value" of $\mathcal{H}$.

It was proved in our former paper [5] that the singular value structure (19) can be characterized by an algebraic equation using L_c and L_o if the target system is an input-affine continuous-time nonlinear system. However, this result were not directly applicable to general (neither non-affine nor discrete-time) nonlinear systems so far.

6. Singular value analysis of Hankel operators

The objective of this section is to provide the solution of Equations (19) and (20) for singular value analysis of the Hankel operator $\mathcal{H}$. Here we assume the smoothness of the operators C, O and $C^\dagger$.

Assumption A1 the operators $C : U \to X$, $O : X \to Y$ and $C^\dagger : X \to U$ exist and are differentiable.

Under Assumption A1 we can obtain an alternative characterization of singular value analysis of the Hankel operator on the signal space X which is much simpler than (19).

First of all, in order to characterize the signal space satisfying the constraint (20), let us consider the properties of the pseudo-inverse operator $C^\dagger$. By Assumption A1, both C and $C^\dagger$ exist and are smooth. Hence the constraint (20) can be characterized by singular value analysis of $C^\dagger \circ C$. That is,

$$\arg\sup_{u\in U} \frac{\|C^\dagger \circ C(u)\|}{\|u\|}$$

with the maximum singular value 1 characterizes the elements of $\mathrm{Im}\, C^\dagger$, because

$$\frac{\|C^\dagger \circ C(u)\|}{\|u\|} = 1 \qquad u \in \mathrm{Im}\, C^\dagger$$

$$\frac{\|C^\dagger \circ C(u)\|}{\|u\|} < 1 \qquad \text{otherwise}$$

hold for the definition of $C^\dagger$ in (18).

By the argument similar to Equation (4) we know that Equation (20) reduces to singular value analysis

$$(\mathrm{d}(C^\dagger \circ C)(v))^* \circ C^\dagger \circ C(v) = \frac{\|C^\dagger \circ C(v)\|}{\|v\|}\, v = v$$

since the maximum singular value is 1. This turns out to be

$$(\mathrm{d}C(v))^* \circ (\mathrm{d}C^\dagger(C(v)))^* \circ C^\dagger \circ C(v) = v. \qquad (22)$$

On the other hand, the decomposition of $\mathcal{H}$ in (13) implies that the singular value analysis equation (19) can be written as

$$(\mathrm{d}C(v))^* \circ (\mathrm{d}O(C(v)))^* \circ O \circ C(v) = \lambda\, v. \qquad (23)$$

Comparing (22) and (23), we obtain a sufficient condition to characterize the singular structure of $\mathcal{H}$ as

$$(\mathrm{d}O(C(v)))^* \circ O \circ C(v) = \lambda\, (\mathrm{d}C^\dagger(C(v)))^* \circ C^\dagger \circ C(v) \quad (24)$$

using the linearity of the operator $(\mathrm{d}C(v))^*$. Defining the intermediate signal $\xi := C(v)$, we can obtain a simpler expression

$$(\mathrm{d}O(\xi))^* \circ (O(\xi)) = \lambda\, (\mathrm{d}C^\dagger(\xi))^* \circ (C^\dagger(\xi)). \qquad (25)$$

Recall that the derivative of the controllability and observability functions L_c and L_o defined in (16) and (17) are given by

$$\mathrm{d}L_c(x)(\mathrm{d}x) = \langle C^\dagger(x), \mathrm{d}C^\dagger(x)(\mathrm{d}x)\rangle = \langle(\mathrm{d}C^\dagger(x))^* \circ (C^\dagger(x)), \mathrm{d}x\rangle \qquad (26)$$

$$\mathrm{d}L_o(x)(\mathrm{d}x) = \langle O(x), \mathrm{d}O(x)(\mathrm{d}x)\rangle = \langle(\mathrm{d}O(x))^* \circ (O(x)), \mathrm{d}x\rangle \qquad (27)$$

Therefore Equation (19) reduces down to

$$\mathrm{d}L_o(\xi) = \lambda\, \mathrm{d}L_c(\xi).$$

Finally we can obtain the following result which is the generalized version of the result in [5] in the sense that it is applicable to a larger class of input-state-output systems.

Theorem 6 *Suppose that Assumption A1 holds. Assume moreover that there exist $\lambda \in \mathbb{R}$ and $\xi \in X$ satisfying*

$$\mathrm{d}L_o(\xi) = \lambda\, \mathrm{d}L_c(\xi). \qquad (28)$$

Then $v \in U$ defined by

$$v := C^\dagger(\xi) \qquad (29)$$

is a singular vector of $\mathcal{H}$.

Note that the corresponding singular value σ defined in (21) is given by

$$\sigma = \frac{\|\mathcal{H}(v)\|}{\|v\|} = \frac{\|O \circ C \circ C^\dagger(\xi)\|}{\|C^\dagger(\xi)\|} = \frac{\|O(\xi)\|}{\|C^\dagger(\xi)\|} = \sqrt{\frac{(1/2)\|O(\xi)\|^2}{(1/2)\|C^\dagger(\xi)\|^2}} = \sqrt{\frac{L_o(\xi)}{L_c(\xi)}}.$$

In particular, if we can characterize all the solutions ξ_i's of (28) and let σ_i's denote the corresponding singular values, then clearly we obtain the property that the Hankel norm, which is the gain of the Hankel operator, coincides with the maximum singular value. That is,

$$\sup_{\substack{u\in U\\ u\neq 0}} \frac{\|\mathcal{H}(u)\|}{\|u\|} = \sup_{\substack{i\\ \xi_i}} \sigma_i|_{\xi=\xi_i}.$$

Example 4 Suppose that our target system is the linear dynamical system given in Example 1. Then the solution of singular value analysis of the corresponding Hankel operator can be characterized by

$$\xi^{\mathrm{T}} Q = \lambda\, \xi^{\mathrm{T}} P^{-1}$$

with the controllability and observability Gramians P and Q, which is equivalent to

$$PQ\,\xi = \lambda\, \xi.$$

That is, ξ is the eigenvector of PQ and the solution set of ξ plays the role of the coordinate axes of the balanced realization. Furthermore, the singular value σ coincides with the Hankel singular values (square root of the eigenvalues of PQ).

Example 5 Suppose that our target system is the dynamical system given in Example 2 or 3. Then the solution of singular value analysis of the corresponding Hankel operator can be characterized by an algebraic equation

$$\frac{\partial L_o}{\partial x}(\xi) = \lambda \frac{\partial L_c}{\partial x}(\xi). \qquad (30)$$

Similarly to the linear case, the solution set of ξ plays the role of the axes of the balanced coordinates. Furthermore, in this case, if the jacobian linearization of the system is controllable, then there exists a coordinate transformation $\bar{x} = \Phi(x)$ such that the transformed system satisfies [11]

$$L_c(\Phi^{-1}(\bar{x})) = \frac{1}{2}\bar{x}^{\mathrm{T}}\bar{x}.$$

Hence the equation (30) further reduces down to

$$\frac{\partial L_o(\Phi^{-1}(\bar{x}))}{\partial \bar{x}}(\bar{\xi}) = \lambda\,\bar{\xi}.$$

Obviously, this is a nonlinear eigenvalue problem with a *differentially self-adjoint* operator. Further, Theorems 4 and 5 are directly applicable to this problem, since our input signal space $X = \mathbb{R}^n$ now.

Please note that we do not require any state-space realization of any operator here. So Theorem 6 is applicable to very general nonlinear systems including both continuous and discontinuous both input-affine and input-non-affine dynamical system.

7. Conclusion

This paper proposed a novel framework for singular value analysis of nonlinear operators. First of all, a natural definition of nonlinear singular values is proposed. Second, it is shown that the singular values thus defined can be obtained by solving a special class of nonlinear eigenvalue problems with respect to differentially self-adjoint operators. Third, some properties of singular values are clarified. A sufficient condition for the existence of singular values is proved. Further, application of the proposed method to singular value analysis of nonlinear Hankel operators exhibits its effectiveness. It is expected that this framework will provide a useful analysis tools for nonlinear control systems theory, in particular for balanced realization and model reduction, as in the linear case.

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