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Stochastic Aspect of Noise-induced Synchronization

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Abstract— We derive a recursion formulae of transition probability of the noise-induced synchronization arising in a pair of identical uncoupled logistic maps linked by common noisy excitation only. The formulae has a delta type stationary solution by which it is explained that the maps can perfectly be synchronized with probability 1.

1. Introduction

One of the most surprising results of the recent years in the field of nonlinear stochastic dynamics is that two identical nonlinear systems can perfectly be synchronized when they share common stochastic or chaotic excitation. This kind of noise-induced synchronization can easily be found in the discrete map and the Lorenz system[1], and the Duffing oscillator[2]. Furthermore, the authors have already shown that the noise-induced synchronization of the van der Pol system can be characterized as the point structure of random invariant measures[3]. In these studies, however, the main interest seems to be in dynamical aspects of the synchronization in view of the chaotic synchronization and related fields[1, 2] or the theory of random dynamical systems[3].

On the contrary, this paper studies a stochastic aspect of the noise-induced synchronization arising in a pair of identical uncoupled logistic maps linked by common noisy excitation only. We regard the response of the mapping as a Markov process and derive the transition law of it. We then analytically show that the Markov process has the absorbing barrier which corresponds to the perfect synchronization. The analytical consideration is in good agreement with the result of Monte Carlo simulations.

2. Noise-induced synchronization

2.1. Synchronization of the logistic maps

We consider a synchronization system composed by a pair of identical uncoupled logistic maps linked by common noisy excitation of the following form:

$$\begin{aligned} x_{n+1} &= A_n x_n(1 - x_n), \\ y_{n+1} &= A_n y_n(1 - y_n) \end{aligned} \tag{1}$$

where A_n is the noisy term which is uniformly distributed in the interval $[A_c - \sigma, A_c + \sigma]$. If $\sigma = 0$, then the system (1) coincides the deterministic logistic map with the constant parameter $A_n = A_c$ for all n. In order to determine the range of σ to be considered, the bifurcation diagram of the one-dimensional logistic map:

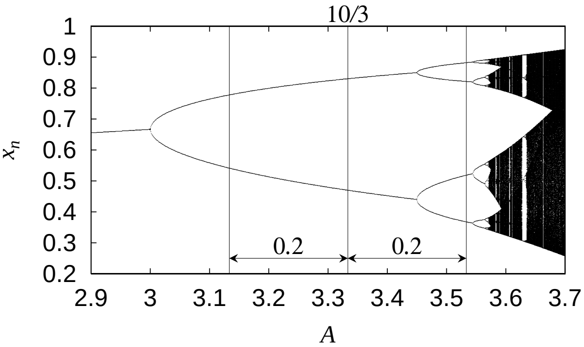

Figure 1: The bifurcation diagram of the logistic map.

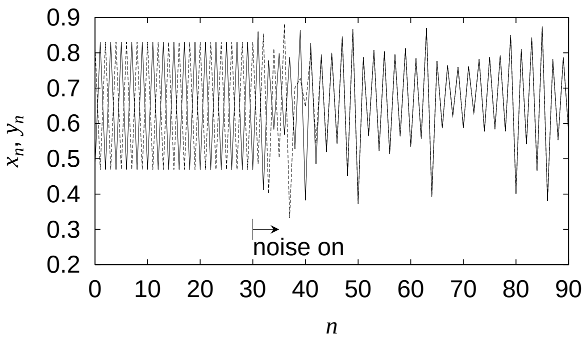

Figure 2: Noise-induced synchronization of the pair of logistic maps linked by common noisy excitation for $\sigma = 0.2$ ($n \geq 30$).

$$x_{n+1} = A x_n(1 - x_n) \tag{2}$$

is shown in Fig 1. From the diagram, we choose the center value $A_c = 10/3$ and strict the value of σ in the range $0 \leq \sigma \leq 0.2$ to avoid the one-periodic domain, $A < A_0 \approx 3$, in which the trivial synchronization of the system (1) occurs.

Fig 2 shows a sample process of the synchronization system (1) whose noise intensity σ is changed from 0 to 0.2 at $n = 30$. In the deterministic case for $n < 30$, the two-periodic responses x_n, y_n of different initial values oscillate with the phase difference of period one. As σ is changed to 0.2 at $n = 30$, the responses x_n, y_n become synchronized with each other.

For further investigation, we introduce the transformation:

$$\begin{bmatrix} r_n \\ s_n \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}.$$

Then, the original equation (1) is rewritten as

$$\begin{aligned} r_{n+1} &= A_n r_n (1 - s_n), \\ s_{n+1} &= A_n \Big(s_n - \frac{r_n^2 + s_n^2}{2} \Big) \end{aligned} \tag{3}$$

where $r_n = x_n - y_n$ represents error of the synchronization. In what follows, we refer to the new equation (3) as an error system of the synchronization system (1).

2.2. Stability of the error system

We first examine the case for the fixed $A_n = A_c$ (for all n). Based on the linearized form of the error system (3) given by

$$\begin{bmatrix} \delta r_{n+1} \\ \delta s_{n+1} \end{bmatrix} = \begin{bmatrix} A_c(1 - s_n) & -A_c r_n \\ -A_c r_n & A_c(1 - s_n) \end{bmatrix} \begin{bmatrix} \delta r_n \\ \delta s_n \end{bmatrix},$$

the stability of the fixed points of (3) is obtained. All the stable fixed points for $A_c = 10/3$ are listed below.

Index	Fixed point	Eigenvalue
FP1	$(0, (13 \pm \sqrt{13})/10)$	$-4/9$
FP2	$(\pm\sqrt{13}/10, \mp\sqrt{13}/10)$	$-4/9$

The trivial (FP1) and the nontrivial (FP2) solutions correspond to the synchronized and the unsynchronized responses of the error system (3) respectively. It is clearly shown that both the synchronized and the unsynchronized solutions have the same eigenvalue $-4/9$ which is a stable eigenvalue of mapping systems because it is placed within the unit circle of the complex plane. Therefore, in the deterministic case, there is no difference in stabilities between the synchronized and the unsynchronized solutions.

By contrast to the deterministic case, the nontrivial solution (FP2) loses its stability in the stochastic case where A_n is random. Fig 3 shows a sample process of the error system (3) starting from the nontrivial solution (FP2). The noise intensity σ is changed from 0 to 0.2 at $n = 30$. In the deterministic case for $n < 30$, the nontrivial solution (FP2) maintains the stability of the stable eigenvalue $-4/9$. However, the nontrivial solution vanishes and jumps into the trivial solution (FP1) as σ is increased to 0.2 at $n = 30$. The trivial solution after $n = 30$ exhibits a strong stability, that is, it is not randomly fluctuated and seems to maintain the constant value $r_n = 0$ while the original system (1) is randomly fluctuated by A_n.

This example makes it clear that the synchronization we consider can not be characterized by the deterministic stability analysis because the difference between the synchronized and the unsynchronized solutions can not be characterized by the same eigenvalue.

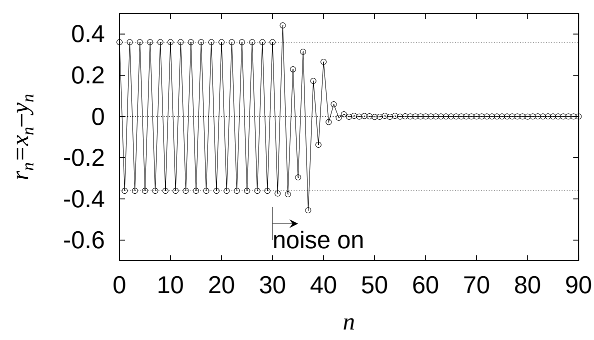

Figure 3: A sample process of the error system for $A = 10/3$ and for $\sigma = 0.2$ ($n \geq 30$).

3. The Markov process generated by the mapping

As another option to characterize the synchronization, we derive a recursion formulae which determines the transition probability densities of the stochastic system (1).

3.1. The single map case

We start with the simplest case, that for the single logistic map with the random coefficient A_n of the form:

$$x_{n+1} = A_n x_n (1 - x_n). \tag{4}$$

Let the probability density function (PDF) of x_n, x_{n+1}, A_n be $p^n(x)$, $p^{n+1}(x)$, $\rho(A)$ respectively, and suppose that $p^n(x)$ is known, $\rho(A)$ is known and stationary, and $p^n(x)$ and $\rho(A)$ are independent. To avoid singularity, we also assume the condition, $0 < x_n < 1$, without loss of generally because the trivial solutions $x = 0, 1$ are not of interest in our investigation. Then, the unknown density $p^{n+1}(x)$ is determined as follows.

We first introduce the transformation: $x_{n+1} = A_n x_n (1 - x_n)$, $y = x_n$, whose Jacobian is given by

$$\frac{\partial(x_{n+1}, y)}{\partial(A^n, x_n)} = x^n(1 - x^n).$$

From the assumption $0 < x < 1$, the transformation is holomorphic so that the unknown joint PDF, $p(x_{n+1}, y)$, can be determined by the known, $p(A_n, x_n) = \rho(A) p^n(x)$, as

$$p(x_{n+1}, y) = \left(\frac{\partial(x_{n+1}, y)}{\partial(A^n, x_n)} \right)^{-1} p^n(x) \rho(A).$$

Integrating it from 0 to 1 with respect to y, the desired $p^{n+1}(x)$ is obtained as the marginal PDF of $p(x_{n+1}, y)$,

$$\begin{aligned} p^{n+1}(x) &= \int_0^1 p(x_{n+1}, y) dy \\ &= \int_0^1 \frac{p^n(y)}{y(1 - y)} \rho \left(\frac{x}{y(1 - y)} \right) dy. \end{aligned} \tag{5}$$

Therefore, the transition law from $p^n(x)$ to $p^{n+1}(x)$ is obtained as the recursion formulae (5) which governs the Markov process generated by the map (4).

3.2. The linked pair case

Such a transformation of PDF also leads to the transition law of the synchronization system (1), however, some additional trick is needed in this case. We thus start with the unlinked form:

$$\begin{aligned} x_{n+1} &= A_n x_n(1-x_n),\\ y_{n+1} &= B_n y_n(1-y_n) \end{aligned} \tag{6}$$

Let the joint probability density of x_n and y_n be $p^n(x,y)$, and that of A_n and B_n be $\rho(A,B)$, and suppose that the response is independent of the input at the same time, i.e., $p(x_n,y_n,A_n,B_n) = p(x_n,y_n)\rho(A,B)$, and that ρ is stationary. Then, we introduce the following transformation:

$$\begin{aligned} x_{n+1} &= A_n x_n(1-x_n),\\ y_{n+1} &= B_n y_n(1-y_n),\\ u &= x_n, \quad v = y_n \end{aligned}$$

whose Jacobian is

$$\frac{\partial(x_{n+1},y_{n+1},u,v)}{\partial(x_n,y_n,A_n,B_n)} = x_n y_n(1-x_n)(1-y_n).$$

It is clear that the transformation is holomorphic in the domain considered, $0 < x_n, y_n < 1$. Therefore,

$$\begin{aligned} & p(x_{n+1},y_{n+1},u,v)\\ &= \frac{p^n(u,v)}{uv(1-u)(1-v)}\rho\left(\frac{x_{n+1}}{u(1-u)},\frac{y_{n+1}}{v(1-v)}\right) \end{aligned}$$

and integrating $p(x_{n+1},y_{n+1},u,v)$ from 0 to 1 with respect to u and v, we obtain the recursion formulae:

$$\begin{aligned} p^{n+1}(x,y) = \int_0^1\int_0^1 & \frac{p^n(u,v)}{uv(1-u)(1-v)}\\ & \times\rho\left(\frac{x}{u(1-u)},\frac{y}{v(1-v)}\right)dudv. \end{aligned} \tag{7}$$

To rewrite the unlinked form (7) to the linked form corresponding to the linked pair of maps (1), we assume the joint density $\rho(A,B)$ of the form:

$$\rho(A,B) := \rho(A)\delta(A-B) = \rho(B)\delta(A-B) \tag{8}$$

where the probability density $\rho(A)$ of A_n and $\rho(B)$ of B_n are assumed to be identical, i.e., $\rho(A) = \rho(B)$, and δ is the Dirac's delta function with the following properties[4]:

(d1) $\delta(-x) = \delta(x)$,

(d2) $f(x)\delta(x-a) = f(a)\delta(x-a)$,

(d3) $\delta(ax) = |a|^{-1}\delta(x)$, more generally,

$$\begin{aligned} & g(x_i) = 0 \quad (i = 1,2,\cdots,n)\\ & \Longrightarrow \delta(g(x)) = \sum_{i=1}^{n}|dg(x_i)/dx|^{-1}\delta(x-x_i). \end{aligned}$$

From the definition (8), the probability of the event $A_n \neq B_n$ equals 0 and the marginal density of it is identical to the density $\rho(A) = \rho(B)$, that is,

$$\int_{-\infty}^{\infty}\rho(A,B)dA = \rho(A) = \rho(B) = \int_{-\infty}^{\infty}\rho(A,B)dB.$$

This means that the value of the random variable A_n is identical to that of B_n with probability 1, and thus the assumption (8) reasonably corresponds to the situation where the pair of maps is linked by the common noise A_n, as defined in (1).

Then, the linked version of (7) is obtained in the form:

$$\begin{aligned} p^{n+1}(x,y) = \int_0^1\int_0^1 & \frac{p^n(u,v)}{uv(1-u)(1-v)}\\ & \times\rho\left(\frac{x}{u(1-u)}\right)\delta\left(\frac{x}{u(1-u)}-\frac{y}{v(1-v)}\right)dudv. \end{aligned} \tag{9}$$

Applying (d1)-(d3) to eliminate the delta function from (9), we finally obtain the recursion formulae:

$$\begin{aligned} p^{n+1}(x,y) = \int_0^1\int_0^1 & \rho\left(\frac{x}{u(1-u)}\right)\\ & \times\frac{p^n\left(\frac{1}{2}(1-\Phi),v\right)+p^n\left(\frac{1}{2}(1+\Phi),v\right)}{y\Phi}dudv \end{aligned} \tag{10}$$

where $\Phi := \Phi(x,y,v) = \sqrt{1-4xv(1-v)/y}$.

The recursion formulae (10) describes the transition law from $p^n(x,y)$ to $p^{n+1}(x,y)$. This means that the error among the synchronization system (1) generates the Markov process governed by (10).

4. Probability densities of the synchronization

4.1. A special solution for the perfect synchronization

We first assume a candidate of a stationary solution of the equation (10) of the following form:

$$p^n(x,y) := \delta(x-y)p^n(x) = \delta(x-y)p^n(y)$$

where $p^n(x)$ is a solution of the equation (5) which is the state probability density of the single map (4) at the time n.

Put, $\alpha := x/y$, and

$$\begin{aligned} g_1 &:= \frac{1}{2}(1-\Phi)-v, \quad g_2 := v-\frac{1}{2}(1+\Phi),\\ p_1^n &:= p^n\left(\frac{1}{2}(1-\Phi),v\right) = \delta(g_1)p^n(v),\\ p_2^n &:= p^n\left(\frac{1}{2}(1+\Phi),v\right) = \delta(g_2)p^n(v), \end{aligned}$$

where $\Phi(x,y,v) = \Phi(\alpha,v) = \sqrt{1-4\alpha v(1-v)}$. Then, zeros of $g_i = g_i(\alpha)$ are obtained as a simple point $\alpha = 1$ $(i = 1,2)$, and the derivative of $g_i(\alpha)$ is

$$g_i'(\alpha) = g'(\alpha) = \frac{2v(1-v)}{\sqrt{1-4\alpha v(1-v)}}.$$

Therefore, from (d1)-(d3), we can rewrite p_1^n, p_2^n as

$$p_i^n = \frac{1}{|g_i'(1)|}\delta(x/y-1)p^n(v) = \frac{|y|}{|g_i'(1)|}\delta(x-y)p^n(v)$$
$$= \frac{y\sqrt{1-4v(1-v)}}{2v(1-v)}\delta(x-y)p^n(v) \quad (i=1,2). \qquad (11)$$

Substituting (11) into (10), we have

$$p^{n+1}(x,y)$$
$$= \delta(x-y)\int_0^1 \rho\left(\frac{y}{v(1-v)}\right)\frac{p^n(v)\sqrt{1-4v(1-v)}\,dv}{v(1-v)\sqrt{1-4v(1-v)\frac{x}{y}}}.$$

Since $\delta(x-y)=0$ holds if $x \neq y$,

$$= \delta(x-y)\left(\int_0^1 \rho\left(\frac{y}{v(1-v)}\right)\frac{p^n(v)\sqrt{1-4v(1-v)}\,dv}{v(1-v)\sqrt{1-4v(1-v)\frac{x}{y}}}\right)_{x=y}$$
$$= \delta(x-y)\int_0^1 \rho\left(\frac{y}{v(1-v)}\right)\frac{p^n(v)}{v(1-v)}dv.$$

From the equation (5), finally we have

$$= \delta(x-y)p^{n+1}(y).$$

Therefore, it is proved that $\delta(x-y)p^n(y)$ is a special solution of the equation (10).

This special solution exactly corresponds to the perfect synchronization of x_n and y_n in the system (1) because from the definition of the delta function, the probability of the event, $r_n = x_n - y_n \neq 0$, equals 0 and the density of unit volume is perfectly concentrated on the line $r_n = x_n - y_n = 0$. In other words, values of the random variables x_n and y_n are perfectly synchronized with probability 1.

4.2. Numerical examples

Fig 4 shows the transient probability densities of the synchronization system (1) obtained by Monte Carlo simulations on 2×10^7 samples of the numerical solution of (1) starting from $(x_0, y_0) = (\frac{13-\sqrt{13}}{20}, \frac{13+\sqrt{13}}{20})$ corresponding to one of the nontrivial solutions (FP2).

As the time n is increased, the initial density concentrated in the initial point (x_0, y_0) becomes diffused around. Meanwhile, a part of diffused density becomes captured by the peak on the line $r_n = x_n - y_n = 0$. The diffused density nearly vanishes until $n = 2000$ and only the peak of the form $\delta(x-y)p^n(y)$ is alive.

This numerical result confirms the analytical result that the synchronization system (1) has the stationary density of the form $\delta(x-y)p^n(y)$.

5. Conclusion

We have demonstrated that the pair of identical uncoupled logistic maps can perfectly be synchronized when they are linked by common noisy excitation and shown that the linked pair (1) generates the Markov process having the special solution of the form $\delta(x-y)p^n(y)$.

From what has been investigated, we can reasonably conclude that the perfect synchronization corresponding to the trivial solution (FP1), $r_n = x_n - y_n = 0$, can be identified as an absorbing barrier of the Markov process. In view of this, the nontrivial solution (FP2) can be regarded as a local minimum of the potential higher than that of the trivial solution confined on the absorbing barrier.

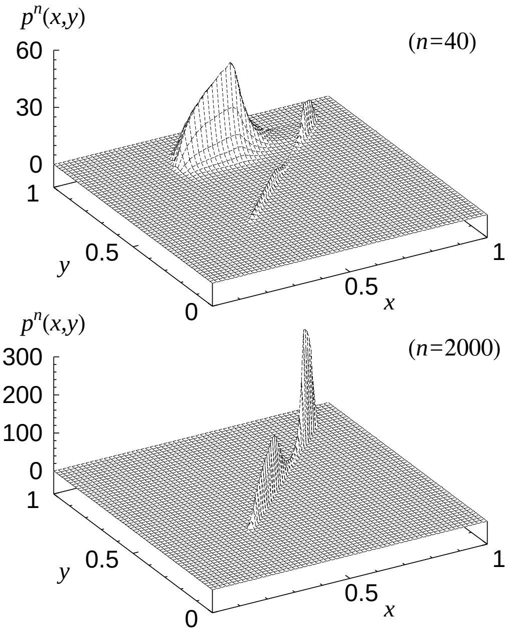

Figure 4: Numerical transient probability densities of the pair of maps for $A = 10/3$, $\sigma = 0.2$ and for $n = 40, 2000$.

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